2. Transmission line theory

Maxwell's eqs. (see p.14 (50)(51), p8 (24))

$$\begin{cases} \nabla \times \mathbb{E} = -j\omega\mu \mathbb{H} - \sigma_m \mathbb{H} = -j\omega \overbrace{\mu\left(1+\frac{\sigma_m}{j\omega\mu}\right)}^{\dot\mu} \mathbb{H} & (70) \\ \nabla \times \mathbb{H} = j\omega\varepsilon \mathbb{E} + \sigma \mathbb{E} = j\omega \underbrace{\varepsilon\left(1+\frac{\sigma}{j\omega\varepsilon}\right)}_{=\dot\varepsilon} \mathbb{E} & (71) \end{cases}$$

wave number

$$k^2 = \omega^2\mu\varepsilon \xrightarrow{\text{replaced by}} \omega^2\mu\varepsilon\left(1-j\frac{\sigma_m}{\omega\mu}\right)\left(1-j\frac{\sigma}{\omega\varepsilon}\right)$$

or.

$$\dot k = \omega\sqrt{\mu\varepsilon}\sqrt{\left(1-j\frac{\sigma_m}{\omega\mu}\right)\left(1-j\frac{\sigma}{\omega\varepsilon}\right)}$$

or, (72)

complex propagation constant : $\dot\gamma \equiv j\dot k = j\omega\sqrt{\mu\varepsilon}\sqrt{\left(1-j\frac{\sigma_m}{\omega\mu}\right)\left(1-j\frac{\sigma}{\omega\varepsilon}\right)}$ $(\alpha+j\beta)$ (73)

Similar discussion in the case of a transmission line

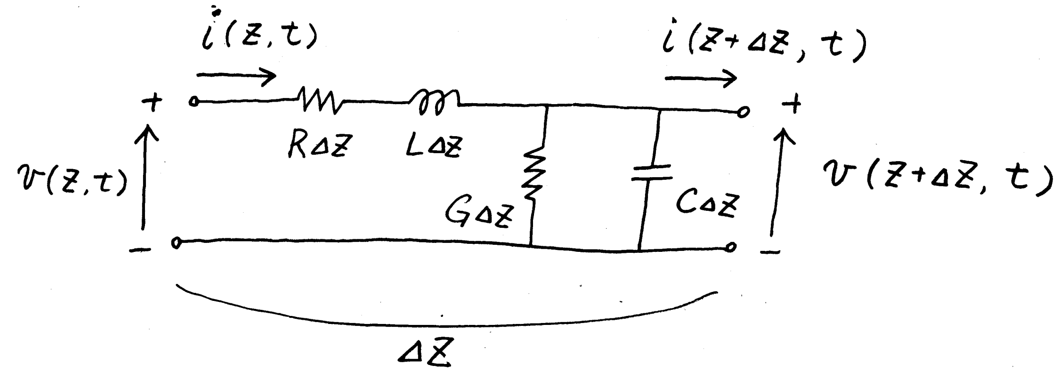


Fig. equivalent circuit for an incremental length of transmission line

Q. What is the relation between v and i ?

$$\begin{cases} \dfrac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L \dfrac{\partial i(z,t)}{\partial t} & (74) \\[2ex] \dfrac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C \dfrac{\partial v(z,t)}{\partial t} & (75) \end{cases}$$

or in the phasor form, $v = \mathrm{Re}[V e^{j\omega t}]$, $i = \mathrm{Re}[I e^{j\omega t}]$

$$\begin{cases} \dfrac{dV(z)}{dz} = -(R + j\omega L)\, I(z) & (76) \\[2ex] \dfrac{dI(z)}{dz} = -(G + j\omega C)\, V(z) & (77) \end{cases}$$

$\Downarrow$ comparison with (70) and (71)

- the complex propagation constant

$$\dot{\gamma} \; (= j\dot{k} = j\sqrt{\omega^2 \dot{\varepsilon}\dot{\mu}}\,) \quad (= \alpha + j\beta)$$

$$= \sqrt{(R + j\omega L)(G + j\omega C)} \qquad (78)$$

equivalent to the 1D Maxwell's equations

- the charasteristic impedance

$$Z_0 \; \left(= \sqrt{\frac{\dot{\mu}}{\dot{\varepsilon}}}\right) = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} \qquad (79)$$

Q. Derive a wave equation from (76) and (77) to get (78).

Ans.
$$\frac{d^2 V}{dz^2} = -(R + j\omega L)\frac{dI}{dz}$$
$$= (R + j\omega L)(G + j\omega C)\, V$$

$$\therefore \quad \frac{d^2 V}{dz^2} - \gamma^2 V = 0, \quad \text{where } \gamma^2 = (R + j\omega L)(G + j\omega C)$$ //

Q. Obtain the propagation constant $\gamma = \alpha + j\beta$ and the characteristic impedance Z_0 for the lossless line ($R = G = 0$). Use (78) and (79).

The solution to (76) and (77) is expressed as

$$\begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} & (80) \\ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} & (81) \end{cases}$$

Q. Apply (76) to (79) to get the current on the line.

$$I(z) = \frac{1}{-(R + j\omega L)} \left[-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} \right]$$

$$= \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}, \qquad (82)$$

where $Z_0 = \dfrac{R + j\omega L}{\gamma}$

$$= \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} \qquad (83)$$

Terminated line

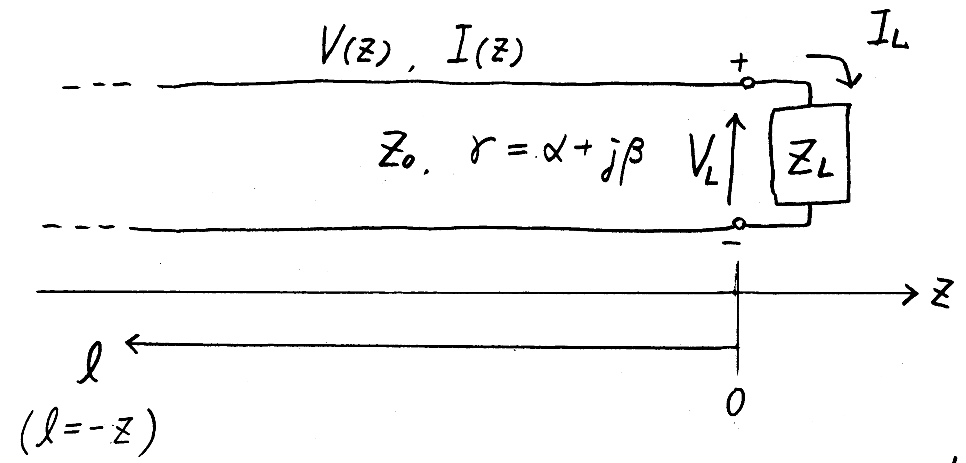


Fig. A terminated transmission line.

The total voltage on the line (sum of incident and reflected wave)

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}, \qquad (84)$$

and the total current: (82)

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \qquad (85)$$

The load (Z_L) condition:

$$Z_L = \frac{V_L}{I_L} = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}}$$

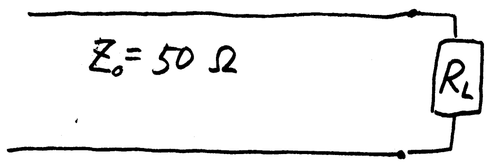
$$= Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \qquad (86)$$

solving (85) for V_0^- gives

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+ \equiv \Gamma V_0^+ \qquad (87)$$

$\Gamma \equiv \frac{V_0^-}{V_0^+}$: voltage reflection ratio

Ex. How large is the reflected voltage in the follwing case ? Also in dB ?



$R_L = 100\ \Omega$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R_L - Z_0}{R_L + Z_0} = \frac{100 - 50}{100 + 50} = \frac{50}{150} = 0.333\cdots \qquad (88)$$

$$[dB] = -20 \log_{10} \Gamma = -20 \times (-0.477) \fallingdotseq 9.54\,dB.$$

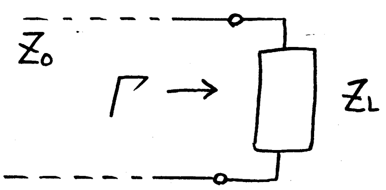
Conversion between Γ and Z_L

From (87)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (87\,a)$$

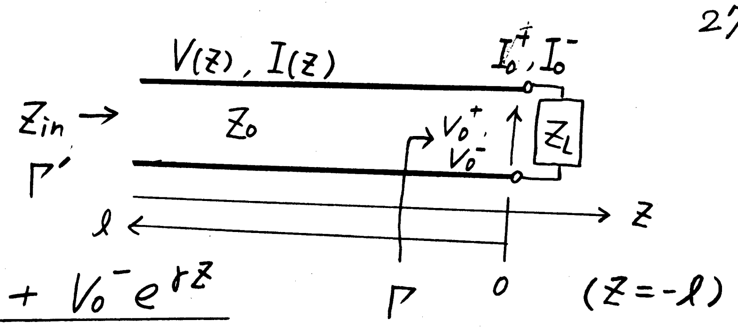
$$\updownarrow \qquad \Gamma(Z_L + Z_0) = Z_L - Z_0$$

$$Z_L = Z_0 \frac{1+\Gamma}{1-\Gamma} \qquad (87\,b)$$



Note : Z_0, Z_L, Γ : complex numbers

Fig (p.25)



$$Z_{in} = \frac{V(z)}{I(z)}$$

$$= \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{\frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}}$$

$$= Z_0 \cdot \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}}$$

$$= Z_0 \cdot \frac{e^{-\gamma z} + (V_0^-/V_0^+) e^{\gamma z}}{e^{-\gamma z} - (V_0^-/V_0^+) e^{\gamma z}}$$

$$= Z_0 \cdot \frac{e^{-\gamma z} + \Gamma e^{\gamma z}}{e^{-\gamma z} - \Gamma e^{\gamma z}} = Z_0 \frac{1 + \Gamma e^{2\gamma z}}{1 - \underbrace{\Gamma e^{2\gamma z}}} \quad (*)$$

$\leftarrow \Gamma' = \Gamma e^{2\gamma z} = \Gamma e^{-2\gamma l}$

Now we know that

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (88)$$

therefore,

$$Z_{in} = Z_0 \frac{(Z_L + Z_0) e^{-\gamma z} + (Z_L - Z_0) e^{\gamma z}}{(Z_L + Z_0) e^{-\gamma z} - (Z_L - Z_0) e^{\gamma z}}$$

$$= Z_0 \frac{Z_L (e^{-\gamma z} + e^{\gamma z}) + Z_0 (e^{-\gamma z} - e^{\gamma z})}{Z_L (e^{-\gamma z} - e^{\gamma z}) + Z_0 (e^{-\gamma z} + e^{\gamma z})} \qquad (89)$$

Since Euler's identity is

$$\begin{cases} e^{\theta} = \cosh\theta + \sinh\theta, \\ e^{-\theta} = \cosh\theta - \sinh\theta, \end{cases} \quad \text{for } \theta\text{: complex number.} \qquad (90)$$

$$Z_{in} = Z_0 \frac{Z_L \cosh\gamma z + Z_0(-\sinh\gamma z)}{Z_L(-\sinh\gamma z) + Z_0 \cosh\gamma z}$$

$$= Z_0 \frac{Z_L - Z_0 \tanh\gamma z}{Z_0 - Z_L \tanh\gamma z} \qquad (\gamma = \alpha + j\beta) \qquad (91)$$

change the variable from z to l, and consider (89) or (91) for lossless case

$$\begin{cases} \gamma = \alpha + j\beta \rightarrow j\beta \quad (\alpha = 0) \\ e^{\gamma z} \rightarrow e^{-\gamma l} \rightarrow e^{-j\beta l} \end{cases}$$

$$Z_{in} = Z_0 \frac{Z_L (e^{j\beta l} + e^{-j\beta l}) + Z_0 (e^{j\beta l} - e^{-j\beta l})}{Z_L (e^{j\beta l} - e^{-j\beta l}) + Z_0 (e^{j\beta l} + e^{-j\beta l})}$$

$$= Z_0 \frac{Z_L (\cos\beta l) + Z_0 (j \sin\beta l)}{Z_L (j \sin\beta l) + Z_0 (\cos\beta l)}$$

$$= Z_0 \frac{Z_L + j Z_0 \tan\beta l}{Z_0 + j Z_L \tan\beta l} \quad // \qquad (92)$$

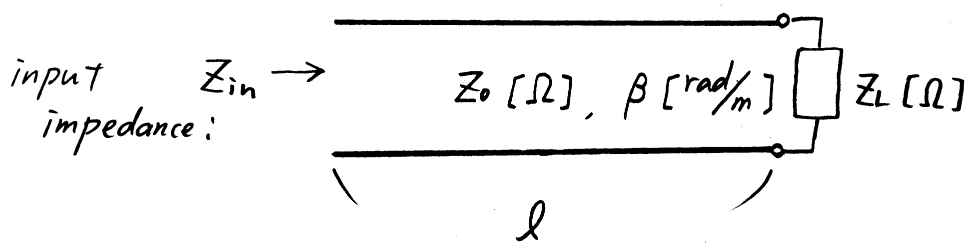


Fig. terminated lossless line

From (92),

short termination ($Z_L = 0$) : $Z_{in} = j Z_0 \tan\beta l$

open termination ($Z_L = \infty$) : $Z_{in} = -j Z_0 \cot\beta l$

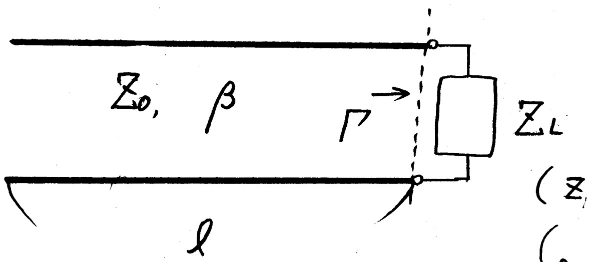
half-wavelength line ($l = \frac{\lambda}{2}$, $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$) : $Z_{in} = Z_L$

quarter-wavelength line ($l = \frac{\lambda}{4}$, $\beta l = \frac{\pi}{2}$) : $Z_{in} = Z_0^2 / Z_L$

$\Downarrow$

quarter-wave transformer

The Smith chart



$\left(z = \dfrac{Z_L}{Z_0} \right.$: normalized impedance)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z - 1}{z + 1} \qquad (93)$$

Suppose the complex numbers :

$$\begin{cases} \Gamma = \Gamma_r + j\,\Gamma_i \\ z = r + j\,x \end{cases} \qquad (94)$$

rewrite (93) as

$$(z+1)\,\Gamma = z - 1$$

$$z\,(\Gamma - 1) = -(\Gamma + 1)$$

$$z = \frac{1+\Gamma}{1-\Gamma} \qquad (95)$$

substitute (94) into (95),

$$r + jx = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} = \frac{\{(1+\Gamma_r)+j\Gamma_i\}\{(1-\Gamma_r)+j\Gamma_i\}}{\{(1-\Gamma_r)-j\Gamma_i\}\{(1-\Gamma_r)+j\Gamma_i\}}$$

$$r = \frac{(1+\Gamma_r)(1-\Gamma_r) - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}, \qquad x = \frac{(1+\Gamma_r)\Gamma_i + \Gamma_i(1-\Gamma_r)}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$= \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}, \quad (96a) \qquad = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}, \quad (96b)$$

From (96a)

$$r(1-\Gamma_r)^2 + r\Gamma_i^2 = 1-\Gamma_r^2-\Gamma_i^2$$

$$r - 2r\Gamma_r + r\Gamma_r^2 + r\Gamma_i^2 = 1-\Gamma_r^2-\Gamma_i^2$$

$$(r+1)\Gamma_r^2 - 2r\Gamma_r + (r+1)\Gamma_i^2 + r-1 = 0$$

$$\Gamma_r^2 - \frac{2r}{r+1}\Gamma_r + \Gamma_i^2 + \frac{r-1}{r+1} = 0$$

$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 - \left(\frac{r}{r+1}\right)^2 + \Gamma_i^2 + \frac{r-1}{r+1} = 0$$

$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 - \frac{r^2}{(r+1)^2} + \frac{r^2-1}{(r+1)^2} = 0$$

$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{1}{(r+1)^2}, \qquad (97a)$$

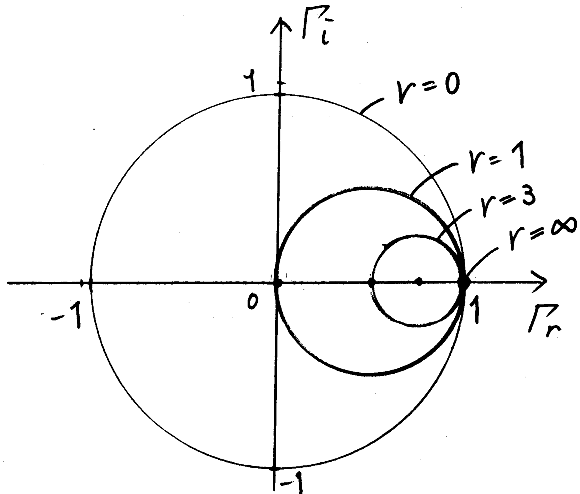
which means a circle with center $(\Gamma_r, \Gamma_i) = \left(\frac{r}{r+1}, 0\right)$, and the radius $\frac{1}{r+1}$, and From (96b)

$$x(1-\Gamma_r)^2 + x\Gamma_i^2 = 2\Gamma_i$$

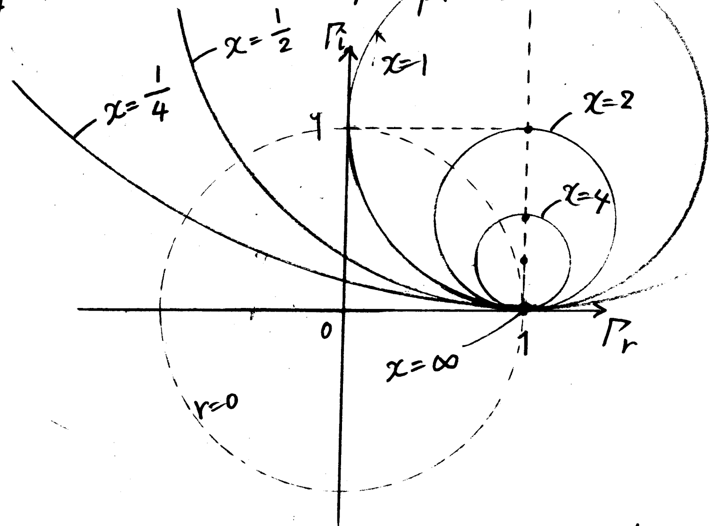
$$(\Gamma_r-1)^2 + \Gamma_i^2 - \frac{2}{x}\Gamma_i = 0$$

$$(\Gamma_r-1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}, \qquad (97b)$$

which means a circle with center $(\Gamma_r, \Gamma_i) = \left(1, \frac{1}{x}\right)$ and the radius $\frac{1}{x}$. Ploting the circles on Γ plane

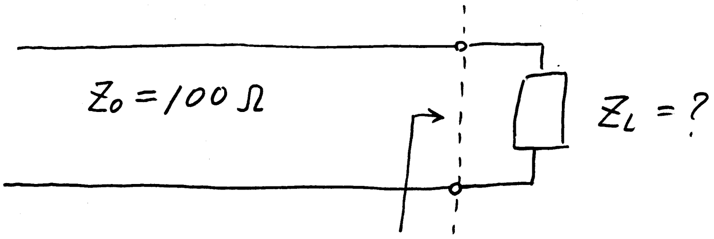


(97a), constant-r circles



(97b) constant-x circles

Ex. 1



$\Gamma = 0.560 + j0.215$

what's the load impedance ?

Ans. $\Gamma = 0.560 + j0.215$

$= \sqrt{0.560^2 + 0.215^2} \angle \tan^{-1} \frac{0.215}{0.560}$

$= 0.6 \angle 21°$

↓ plot on Smith chart,

$r = 2.6$, $x = 1.8$

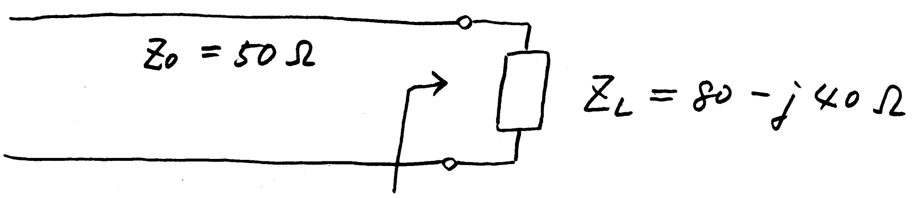
↓

$z_L = 2.6 + j1.8$ (normalized impedance)

↓

$Z_L = Z_0 z_L = 100 \cdot (2.6 + j1.8) = 260 + j180\ \Omega$

Ex. 2



- $\Gamma = ?$,
- return loss [dB] = ?
- Standing wave ratio = ?

Ans.

$z_L = \frac{Z_L}{Z_0} = 1.6 - j0.8$

↓

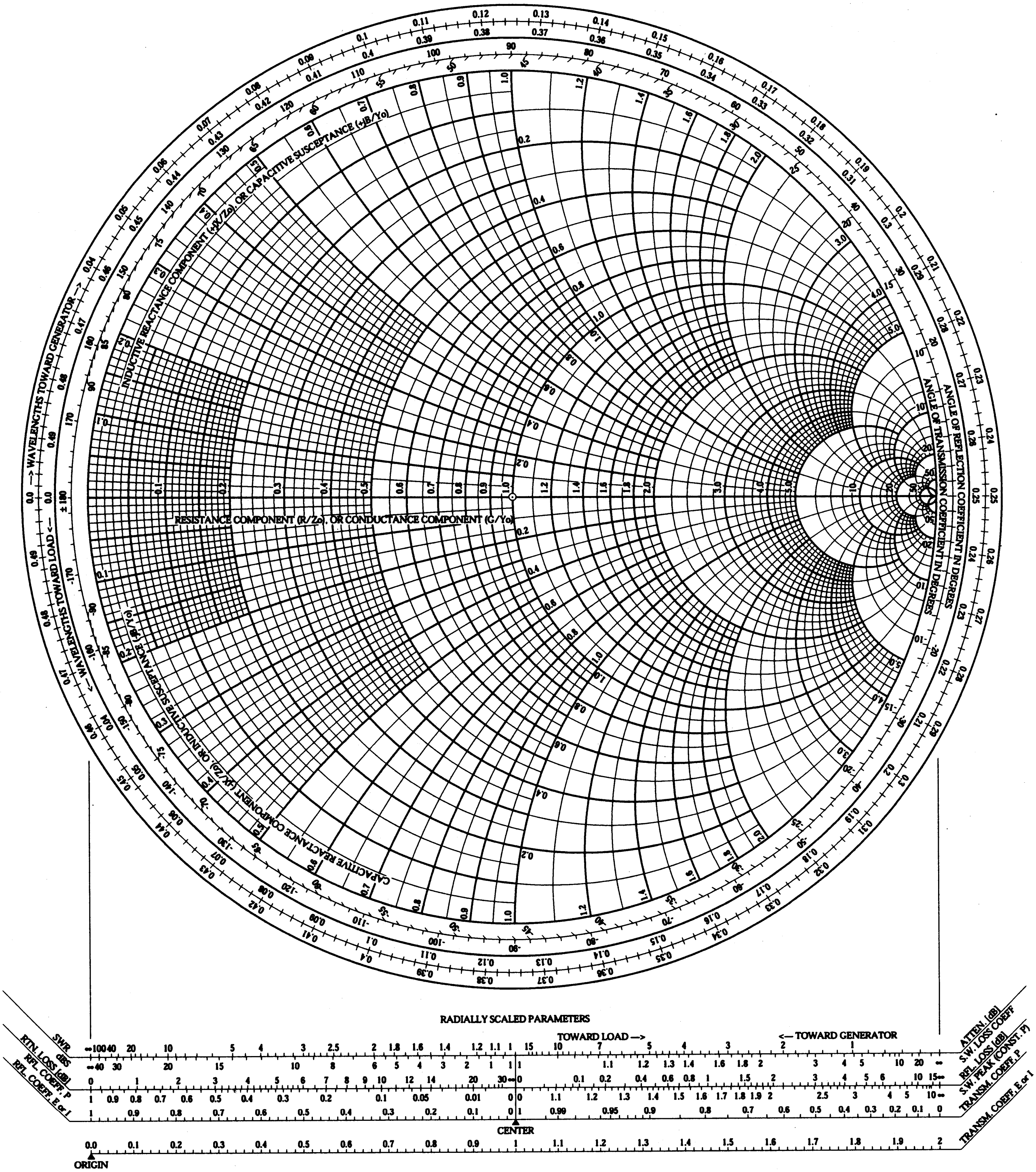
From Smith chart, we read that

$\Gamma = 0.36 \angle -36°$

return loss = 8.7 dB, SWR = 2.1

The Complete Smith Chart

Black Magic Design



The Complete Smith Chart

Black Magic Design

Ex. 1

$\Gamma = 0.560 + j0.215$

$= \sqrt{0.560^2 + 0.215^2}$

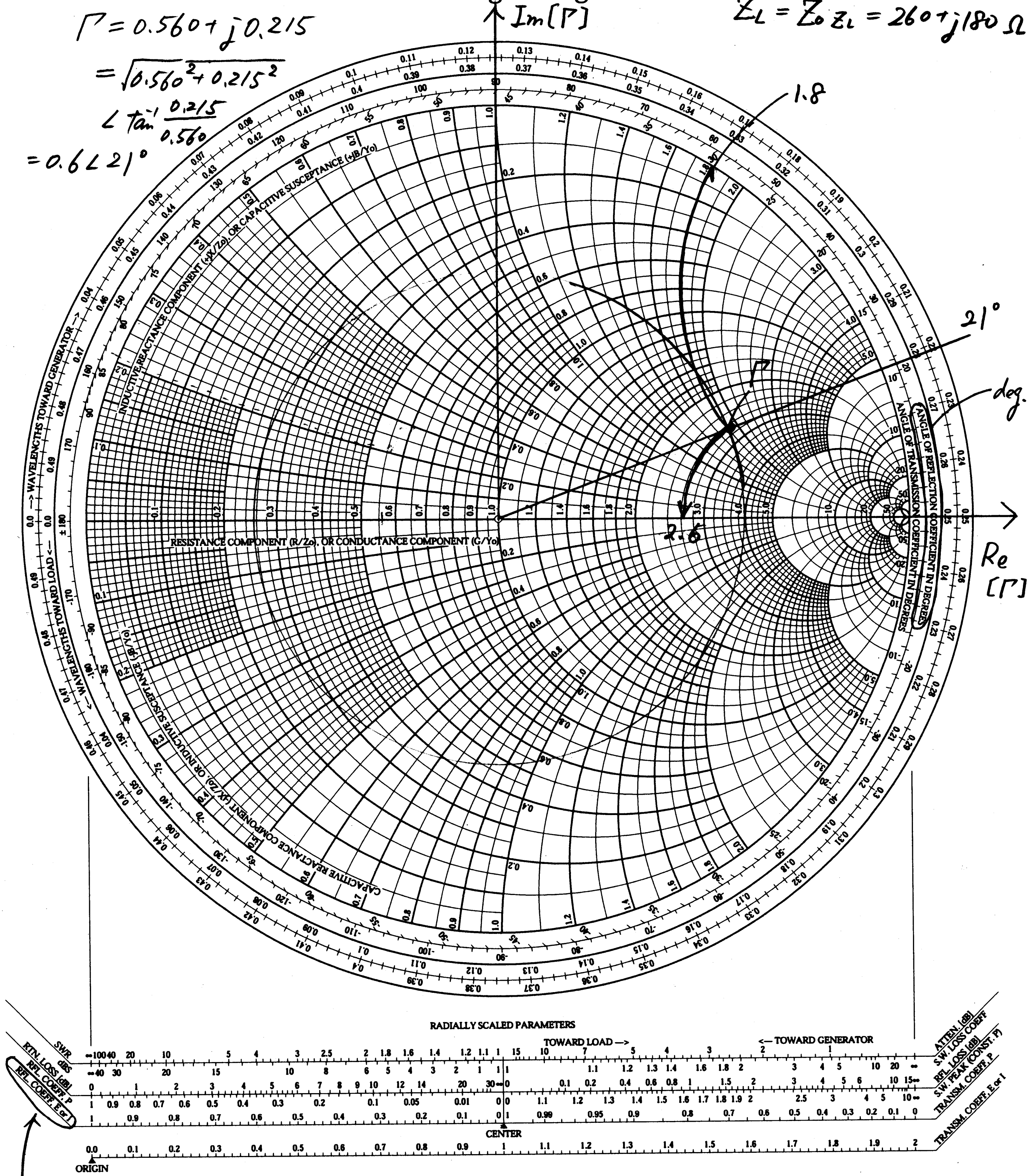
$\angle \tan^{-1}\frac{0.215}{0.560}$

$= 0.6\angle 21°$

$Z_0 = 100\,\Omega,$

$z_L = 2.6 + j1.8$

$Z_L = Z_0 z_L = 260 + j180\,\Omega$



Voltage reflection coeff. Γ
(Current)

Ex. 2

The Complete Smith Chart

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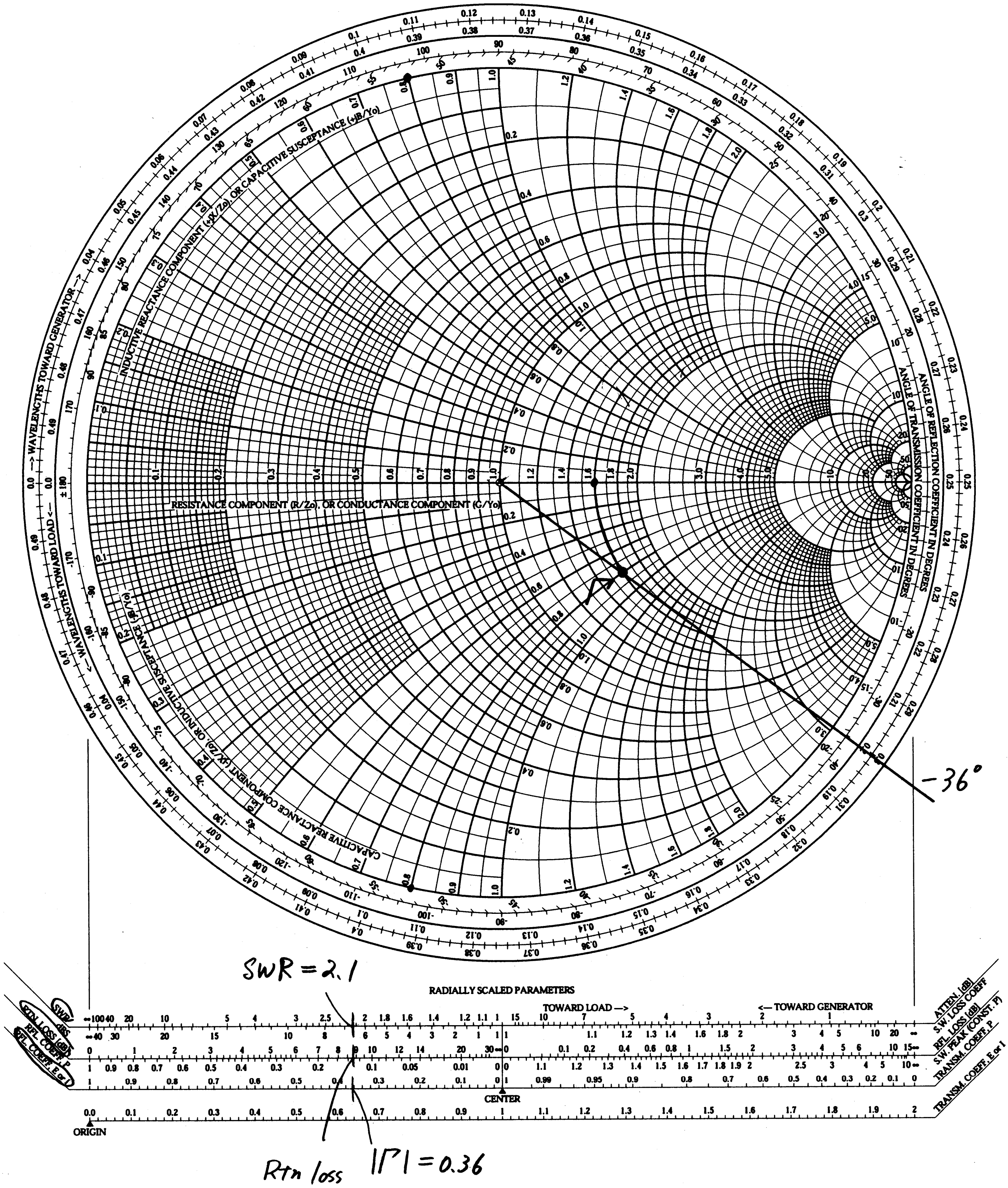
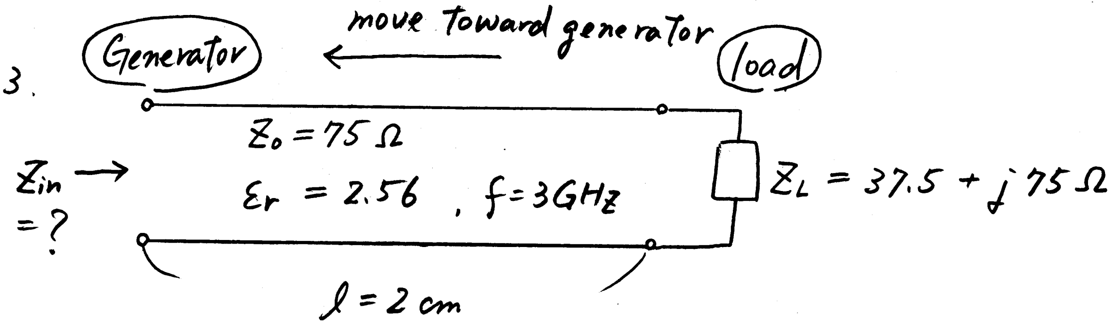


Ex. 3.



SWR = ?

Ans. $\quad z = \dfrac{Z_L}{Z_0} = 0.5 + j \cdot 1$

At 3GHz with $\varepsilon_r = 2.56$,

$$\lambda = \frac{c_0}{\sqrt{\varepsilon_r}\, f} = \frac{3 \times 10^8}{\sqrt{2.56} \cdot 3 \times 10^9} = 0.0625\ m$$

$$l = 2\,cm = 0.32\,\lambda$$

plot.

move toward generator.

$$z_{in} = 0.25 - j\,0.27$$

$$Z_{in} = Z_0\, z_{in} = 75\,(0.25 - j\,0.27) = 18.75 - j\,20.25\,\Omega$$

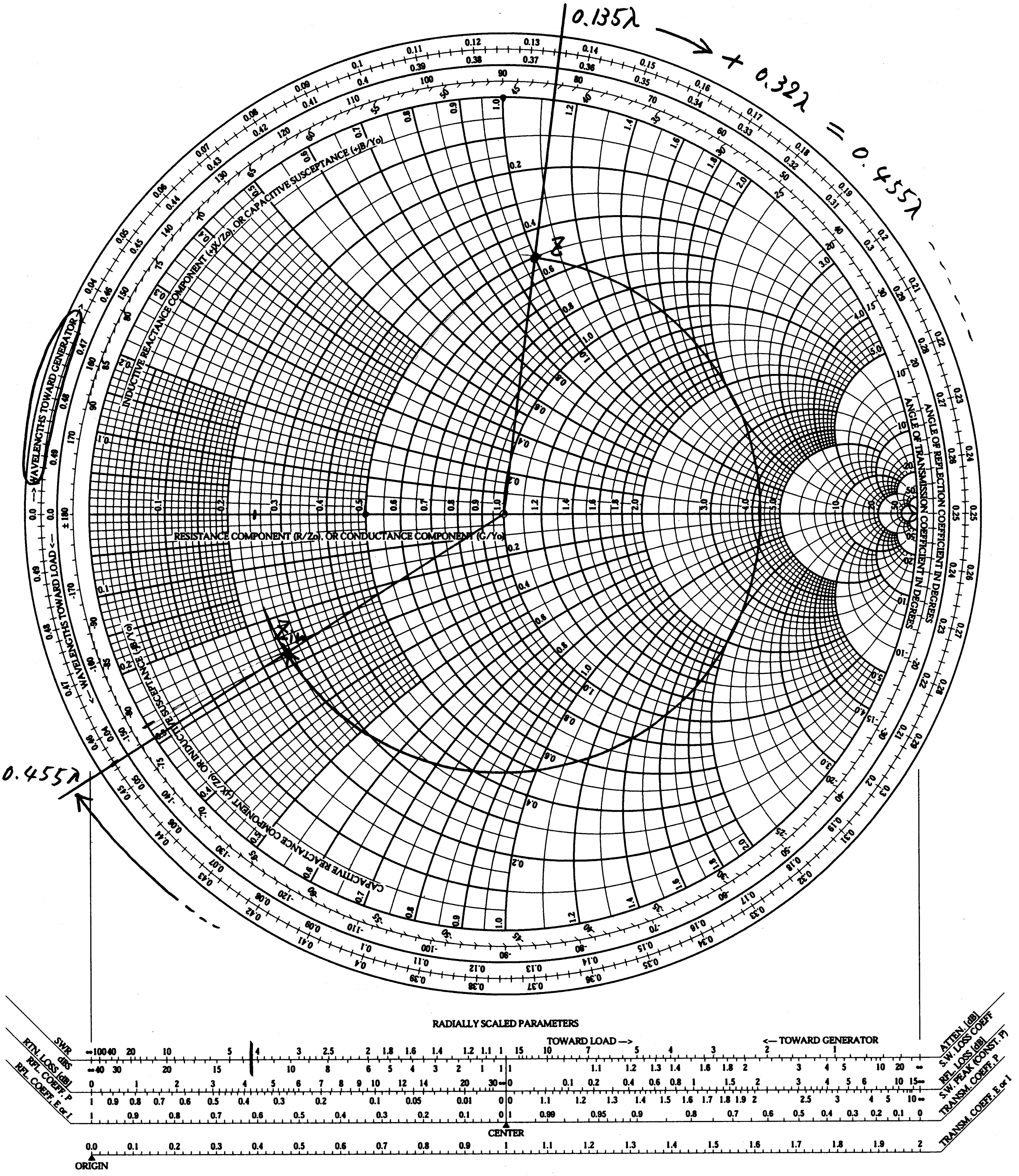
SWR ~ 4.2 (from scale)

Ex. 3.

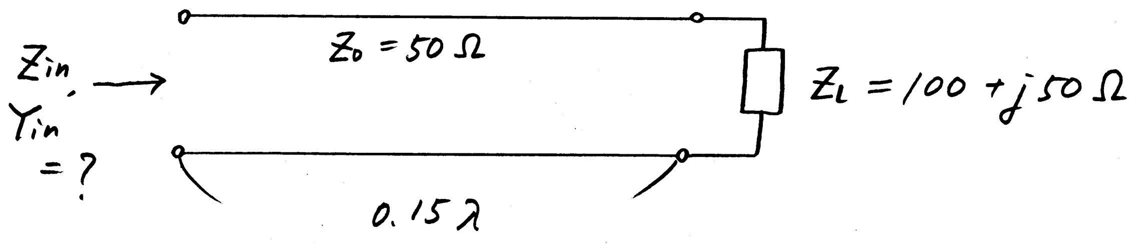
The Complete Smith Chart

Black Magic Design

0.135λ → + 0.32λ = 0.455λ



Ex. 4



Ans. $z_L = \dfrac{Z_L}{Z_0} = 2 + j1$

↓ plot.

move toward generator by 0.15λ

↓

$z_{in} = 0.75 - j0.81$

$Z_{in} = Z_0 z_{in} = 37.5 - j40.5\,\Omega$

superimpose the admittance chart and read that

$y_{in} = 0.62 + j0.67$

↑ attention!

$Y_{in} = Y_0 y_{in} = \dfrac{1}{Z_0} y_{in} = \dfrac{0.62 + j0.67}{50}$

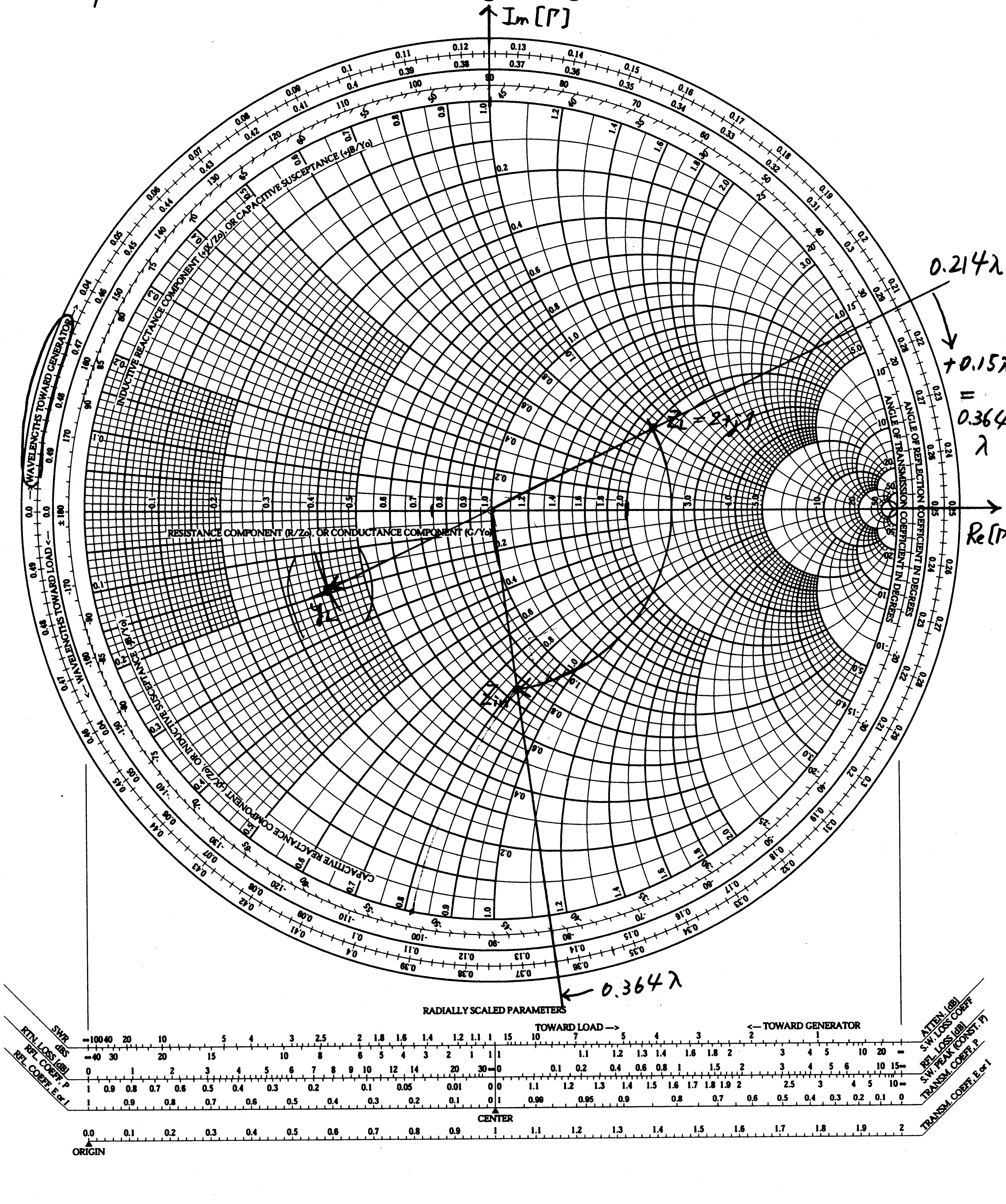
$= 0.0124 + j0.0134$ S

Ex. 4

impedance scale

The Complete Smith Chart

Black Magic Design



Ex. 4 admittance scale (180° ratation of impedance scale)