A proof of Matsumoto-type theorem of (non-crystallographic) Coxeter groups

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1 Semigroup, Monoid, Group

Let K be a set. Assume $K \neq \emptyset$. Let $\lambda : K \times K \to K$ be a map. For x, $y \in K$, denote $\lambda(x, y)$ by xy. We call (K, f) a *semigroup* if $\forall x$, $\forall y$, $\forall z \in K$, $(xy)z = x(yz)$ (this means $\lambda(\lambda(x, y), z) = \lambda(x, \lambda(y, z))$. We also denote (K, f) by K for simplicity. If K and K' are semigroups.

2 Basic representation ρ of the Coxeter group W

For a, $b \in \mathbb{R}$, let $J_{a,b} :=\in \{z \in \mathbb{Z} | a \leq z \leq b\}$. For $a \in \mathbb{Z}$, let $J_{a,\infty} := \{z \in \mathbb{Z} | a \leq z\}$.

Fix $N \in \mathbb{N}$. Let $I := J_{1,N}$. Let $M = [m_{ij}]_{i,j \in I}$ be an $N \times N$ matrix with $m_{ij} \in \mathbb{N} \cup \{+\infty\}$. We call M a *Coxeter matrix* if $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ $(i \neq j)$. Let $W = W(M) := \langle\, s_i \,(i \in I) \,|\, (s_i s_j)^{m_{ij}} = e \,(i \neq j,\, m_{ij} < +\infty) \,\rangle$ be the *Coxeter group* of type M.

Define the map $\ell : W \to \mathbb{Z}_{\geq 0}$ by

$$(2.1) \qquad \ell(w) := \begin{cases} 0 & \text{if } w = e, \\ \min\{n \in \mathbb{N} \,|\, \exists i_x \in I \,(x \in J_{1,n}),\, w = s_{i_1} \cdots s_{i_n} \} & \text{otherwise.} \end{cases}$$

Define the group homomorphism $\mathrm{sgn} : W \to \{\pm 1\}$ by $\mathrm{sgn}(s_i) := 1$. Then $\mathrm{sgn}(w) = (-1)^{\ell(w)}$.

Lemma 2.1. *$\forall w \in W$, $\forall i \in I$, $|\ell(s_i w) - \ell(w)| = 1$.*

Proof. Clearly $\ell(s_i w) \leq \ell(w) + 1$, and $\ell(w) \leq \ell(s_i w) + 1$. Hence $|\ell(s_i w) - \ell(w)| \leq 1$. Since $\mathrm{sgn}(s_i w) \neq \mathrm{sgn}(w)$, we have $|\ell(s_i w) - \ell(w)| \neq 0$. □

Let V be an N-dimensional $\mathbb{R}$-linear space with a basis $\{v_i | i \in I\}$. Let $(\ ,\ ) : V \times V \to \mathbb{R}$ be a bi-linear map defined by

$$(2.2) \qquad (v_i, v_j) := \begin{cases} -2\cos(\pi/m_{ij}) & \text{if } m_{ij} < +\infty, \\ -2 & \text{if } m_{ij} = +\infty. \end{cases}$$

For a subspace V' of V, let $(V')^{\perp} := \{x \in V | \forall y \in V',\, (x, y) = 0\}$. Since $(v_i, v_i) = 2 \neq 0$, we have $V = \mathbb{R}v_i \oplus (\mathbb{R}v_i)^{\perp}$.

Lemma 2.2. *We have a group homomorphism* $\rho : W \to \mathrm{GL}(V)$ *defined by*

$$\rho(s_i)(x) := x - (x, v_i)v_i. \tag{2.3}$$

Proof. We may assume $i \neq j$ and $m_{ij} < +\infty$. Then $0 < \cos(\pi/m_{ij}) < 1$, $0 < \sin(\pi/m_{ij}) < 1$ and

$$\det \begin{bmatrix} 2 & -2\cos(\pi/m_{ij}) \\ -2\cos(\pi/m_{ij}) & 2 \end{bmatrix} = 4\sin^2(\pi/m_{ij}) \neq 0.$$

Hence $V = \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus (\mathbb{R}v_1 \oplus \mathbb{R}v_2)^\perp$. Let $v'_j := (\cos(\pi/m_{ij}))^{-1}(\sin(\pi/m_{ij})v_i + v_j)$. Let $c_1 := \cos(\pi/m_{ij})$ and $c_2 := \sqrt{1-c_1^2}$, so $\sin(\pi/m_{ij}) = c_2$. We have

$$\begin{aligned}
&[\rho_i\rho_j(v_i), \rho_i\rho_j(v'_j)] = [\rho_i\rho_j(v_i), \rho_i\rho_j(v_j)] \begin{bmatrix} 1 & c_1/c_2 \\ 0 & 1/c_2 \end{bmatrix} \\
&= [\rho_i(v_i), \rho_i(v_j)] \begin{bmatrix} 1 & 0 \\ 2c_1 & -1 \end{bmatrix} \begin{bmatrix} 1 & c_1/c_2 \\ 0 & 1/c_2 \end{bmatrix} \\
&= [\rho_i(v_i), \rho_i(v'_j)] \begin{bmatrix} 1 & -c_1 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 & c_1/c_2 \\ 2c_1 & (2c_1^2-1)/c_2 \end{bmatrix} \\
&= [\rho_i(v_i), \rho_i(v'_j)] \begin{bmatrix} 1-2c_1^2 & 2c_1c_2 \\ 2c_1c_2 & 2c_1^2-1 \end{bmatrix} \\
&= [v_i, v'_j] \begin{bmatrix} 1 & -c_1 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -1 & 2c_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c_1/c_2 \\ 0 & 1/c_2 \end{bmatrix} \begin{bmatrix} 1-2c_1^2 & 2c_1c_2 \\ 2c_1c_2 & 2c_1^2-1 \end{bmatrix} \\
&= [v_i, v'_j] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-2c_1^2 & 2c_1c_2 \\ 2c_1c_2 & 2c_1^2-1 \end{bmatrix} \\
&= [v_i, v'_j] \begin{bmatrix} 2c_1^2-1 & -2c_1c_2 \\ 2c_1c_2 & 2c_1^2-1 \end{bmatrix} \\
&= [v_i, v'_j] \begin{bmatrix} \cos(2\pi/m_{ij}) & -\sin(2\pi/m_{ij}) \\ \sin(2\pi/m_{ij}) & \cos(2\pi/m_{ij}) \end{bmatrix}.
\end{aligned} \tag{2.4}$$

For i, $j \in I$ with $i \neq j$, let $W_{i,j}$ be the subgroup of W generated by s_i and s_j, and define $\ell_{ij} : W_{ij} \to \mathbb{Z}_{\geq 0}$ by

$$\ell_{ij}(w_{ij}) := \begin{cases} 0 & \text{if } w_{ij} = e, \\ \min\{n \in \mathbb{N} \,|\, \exists i_x \in \{i,j\}\,(x \in J_{1,n}),\, w = s_{i_1} \cdots s_{i_n}\} & \text{otherwise.} \end{cases} \tag{2.5}$$

Lemma 2.3. *For* i, $j \in I$ *with* $i \neq j$, *we have*

$$m_{ij} = \min\{m \in \mathbb{N} \cup \{+\infty\} | \rho(s_is_j)^m = \mathrm{id}_V\},$$

and $|W_{ij}| = 2m_{ij}$. *(Note that at this moment,* $\ell_{|W_{ij}}$ *may differ from* ℓ_{ij}*.)*

3 Ordinary transformations

Let V be a finite dimensional $\mathbb{R}$-linear space.

Let $N \in \mathbb{N}$, and $I := J_{1,N}$. Let $f_i \in V^* \setminus \{0\}$ for $i \in I$. Let $H_i := \ker f_i$. Let $A_i := \{y \in V | f_i(y) > 0\}$. Let $A := \cap_{i\in I} A_i$. Let $\rho_i \in \mathrm{GL}(V)$ be such that $\rho_i^2 = \mathrm{id}_V$, and $\ker(\mathrm{id}_V - \rho_i) = \ker f_i$.

Lemma 3.1. $\rho_i(A_i) = \{y \in V | f_i(y) < 0\}$.

Proof. Let $n := \dim V$. There is a basis $\{v_j | j \in J_{1,n}\}$ of V such that $\{v_{j'} | j' \in J_{1,n-1}\}$ is a basis of $\ker f_i$ and $f_i(v_n) > 0$. Then $\rho_i(v_{j'}) = v_{j'}$ ($j' \in J_{1,n-1}$) and $\rho_i(v_n) = \sum_{k\in J_{1,n}} a_k v_k$ for some $a_k \in \mathbb{R}$ ($k \in J_{1,n}$). We have $\rho_i(v_n) + v_n \in \ker f_i$. Hence $a_i = -1$. Then we can see that this lemma holds. □

For $i, j \in I$ with $i \neq j$, let $A_{i,j} := A_i \cap A_j$, let

$$(3.1)\quad m_{i,j} := \begin{cases} +\infty & \text{if } (\rho_i\rho_j)^m \neq \mathrm{id}_V \text{ for all } m \in \mathbb{N}, \\ \min\{m \in \mathbb{N} \,|\, (\rho_i\rho_j)^m = \mathrm{id}_V\} & \text{otherwise.} \end{cases}$$

Let $M := [m_{i,j}]_{i,j\in I}$. Let $\gamma : W(M) \to \mathrm{GL}(V)$ be the group homomorphism defined by $\gamma(s_i) := \rho_i$. Note that $|\rho(W_{i,j})| = 2m_{i,j}$.

Theorem 3.2. *Assume that $A \neq \emptyset$. Assume that $\forall i, \forall j \in I$ with $i \neq j$, $H_i \neq H_j$, $\forall w_{ij} \in \gamma(W_{i,j}) \setminus \{e\}$, $\gamma(w_{ij})(A_{ij}) \cap A_{ij} = \emptyset$.*

(1) *$\forall w \in W \setminus \{e\}$, $\gamma(w)(A) \cap A = \emptyset$. In particular, γ is injective.*

(2) *Let $i \in I$. Let $w \in W$. Then either $\gamma(w)(A) \subset A_i$ or $\gamma(w)(A) \subset \gamma(s_i)(A_i)$ holds. Moreover, $\gamma(w)(A) \subset A_i \Leftrightarrow \ell(s_i w) = \ell(w) + 1$.*

(3) *Let $i, j \in I$ be such that $i \neq j$. Let $w \in W$. Then $\forall w \in W$, $\exists w_{ij} \in W_{ij}$, $\gamma(w)(A) \subset \gamma(w_{ij})(A_{ij})$, $\ell(w) = \ell(w_{ij}^{-1} w) + \ell_{ij}(w_{ij})$.*

Proof. We shall show the following (P_q) and (Q_q) ($q \in \mathbb{Z}_{\geq 0}$) by induction on q.

(P_q): $\forall i \in I$, $\forall w \in W$ with $\ell(w) \leq q$, either of the following $(P_q)_+$ and $(P_q)_-$ holds. $(P_q)_+$: $\gamma(w)(A) \subset A_i$, $(P_q)_-$: $\gamma(w)(A) \subset \gamma(s_i)(A_i)$ and $\ell(s_i w) = \ell(w) - 1$.

(Q_q): $\forall i, \forall j \in I$ with $i \neq j$, $\forall w \in W$ with $\ell(w) \leq q$, $\exists w_{ij} \in W_{ij}$ s.t. $\gamma(w)(A) \subset \gamma(w_{ij})(A_{ij})$, $\ell(w) = \ell(w_{ij}^{-1} w) + \ell_{ij}(w_{ij})$.

Note that (P_0) and (Q_0) hold (let $w_{ij} := e$), since $\ell(w) = 0 \Rightarrow w = e$.

Since $s_i(s_i w) = w$, we see that

$$(3.2)\qquad (P_q) \text{ holds, and } \ell(w) - 1 = \ell(s_i w) = q \Rightarrow \gamma(s_i w)(A) \subset A_i.$$

In this paragraph, we show that this theorem holds under the assumption that the (P_∞) holds.

(P_∞): $\forall i \in I$, $\forall w \in W$, either of the following $(P_\infty)_+$ and $(P_\infty)_-$ holds. $(P_\infty)_+$: $\gamma(w)(A) \subset A_i$, $(P_\infty)_-$: $\gamma(w)(A) \subset \gamma(s_i)(A_i)$ and $\ell(s_i w) = \ell(w) - 1$.

Let $w \in W$ be such that $\gamma(w)(A) \cap A \neq \emptyset$. Let $i \in I$. Then $A_i \cap \gamma(w)(A) \neq \emptyset$. By (P_∞), $\gamma(w)(A) \subset A_i$. Hence $\gamma(w)(A) \subset A$. Since $\gamma(w^{-1})(A) \cap A \neq \emptyset$. We also have $\gamma(w^{-1})(A) \subset A$. Hence $\gamma(w)(A) = A$. For all $j \in I$, since $\gamma(s_j w)(A) = \gamma(s_j)(A) \subset \gamma(s_j)(A_j)$, by (P_∞), we have $\ell(s_j w) = \ell(w) + 1$. Hence $w = e$, and we also see that $\ker\gamma = \{e\}$, as desired.

Step 1. If $|I| = 2$, (P_∞) holds.

Assume $|I| = 2$. Then $A = A_1 \cap A_2$, and γ is injective. Let $n = \dim V$. Then $\dim H_i = n - 1$ $(i \in I)$. Since $H_1 \cap H_2 \neq H_1$, we have $H_1 + H_2 = V$. Since $n = 2(n-1) - \dim(H_1 \cap H_2)$, $\dim(H_1 \cap H_2) = n - 2$. Hence there exists a basis $\{ v_k \mid k \in J_{1,n} \}$ such that $v_{k'} \in H_1 \cap H_2$ $(k' \in J_{3,n})$, $v_1 \in H_1 \setminus H_2$, $f_1(v_1) > 0$ and $v_2 \in H_2 \setminus H_1$, $f_1(v_2) > 0$. Note that $A_1 = \mathbb{R}_{>0} v_1 \oplus H_1$, $A_2 = \mathbb{R}_{>0} v_2 \oplus H_2$, and $A = \mathbb{R}_{>0} v_1 \oplus \mathbb{R}_{>0} v_2 \oplus (H_1 \cap H_2)$. Note that $\forall w \in W$, $\forall v \in H_1 \cap H_2$, $\gamma(w)(v) = v$. Note that

$$\forall i \in I,\ \forall w \in W,\ \gamma(w)(A) \cap \mathbb{R} v_i = \emptyset. \tag{3.3}$$

It follows from the proof of Lemma 3.1, that $\gamma(s_i)(v_{k'}) = v_{k'}$ $(i \in I,\ k' \in J_{3,n})$,

$$\exists b \in \mathbb{R},\ [\gamma(s_1)(v_1), \gamma(s_1)(v_2)] = [v_1, v_2] \cdot \begin{bmatrix} -1 & 0 \\ b & 1 \end{bmatrix}, \tag{3.4}$$

and

$$\exists c \in \mathbb{R},\ [\gamma(s_2)(v_1), \gamma(s_2)(v_2)] = [v_1, v_2] \cdot \begin{bmatrix} 1 & c \\ 0 & -1 \end{bmatrix}. \tag{3.5}$$

Since $\gamma(s_1)(v_1 - b v_2) = -v_1$, by (3.3), we have $b \geq 0$. Likewise, $c \geq 0$. Assume $b = 0$. Since $\gamma(s_1 s_2)(c v_1 + v_2) = -v_2$, we have $c = 0$, so we can easily see that (P_∞) holds. Assume that $b > 0$ and $c > 0$. Then we may assume $b = c$. Let $V' := \mathbb{R} v_1 \oplus \mathbb{R} v_2$. Let $A' := A \cap V' = \mathbb{R}_{>0} v_1 \oplus \mathbb{R}_{>0} v_2$. Identify V', v_1, v_2 with $\mathbb{R}^2$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. Let $w \in W$. Note that $\gamma(w)(A') = \mathbb{R}_{>0}\gamma(w)(v_1) \oplus \mathbb{R}_{>0}\gamma(w)(v_2)$. Since $\gamma(s_1)(v_2) = v_2$, $\gamma(w s_1)(A')$ is left (resp. right) neighborhood of $\gamma(w)(A')$ with the boundary $\mathbb{R}_{>0}\gamma(w)(v_2)$ if $\mathrm{sgn}(w) = 1$ (resp. $\mathrm{sgn}(w) = -1$). Since $\gamma(s_2)(v_1) = v_1$, $\gamma(w s_2)(A')$ is right (resp. left) neighborhood of $\gamma(w)(A')$ with the boundary $\mathbb{R}_{>0}\gamma(w)(v_1)$ if $\mathrm{sgn}(w) = 1$ (resp. $\mathrm{sgn}(w) = -1$). Let $A_1'^{,0} := A_2'^{,0} := A'$, and $A_1'^{,m} := \gamma(s_1)(A_2'^{,m-1})$, $A_2'^{,m} := \gamma(s_2)(A_1'^{,m-1})$ for $m \in \mathbb{N}$. Let $A_i' := A_i \cap V'$ $(i \in I)$. Note $V' = A_i' \cup \mathbb{R} v_i \cup \gamma(s_i)(A_i')$ (disjoit union). We can see that

$$|W| = \infty \ \Leftrightarrow \bigcup_{m=0}^{\infty} A_1'^{,m} \subset A_1 \ \Leftrightarrow \bigcup_{m=0}^{\infty} A_2'^{,m} \subset A_2. \tag{3.6}$$

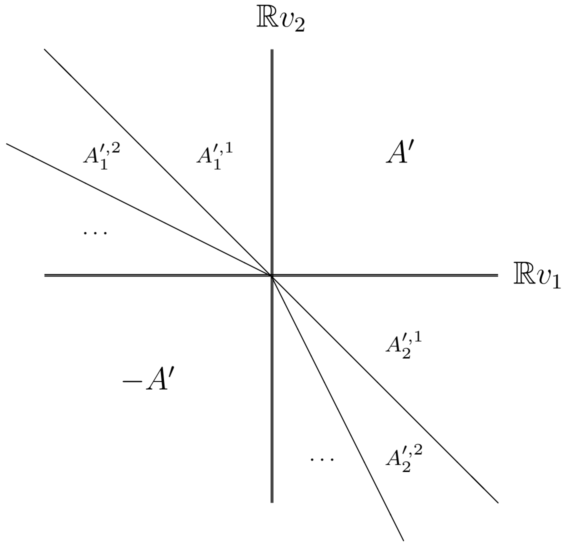

Figure 1: Basic domains of $\gamma(W)$ with $|I| = 2$

Hence, if $|W| = \infty$, we can see that (P_∞) holds.

Assume $|W| < \infty$. Then $|W| = 2m_{12}$. Let $X := \begin{bmatrix} -1 & 0 \\ b & 1 \end{bmatrix}$, and let $Y := \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$. Let $E := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $T := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that $T^2 = E$, and $TXT = Y$. Hence $T(A_k'^{,1}) = A_k'^{,2}$ for all $k \in \mathbb{Z}_{\geq 0}$.

Assume $m_{12} \in 2\mathbb{N}$. Since $\det(XY) = 1$, $(XY)^{m_{12}} = E$ and $(XY)^{m_{12}/2} \neq E$, we have $(XY)^{m_{12}/2} = -E$. Hence $(XY)^{m_{12}/2}(A') = -A'$. Since $|W| = 2m_{12}$, we have $-A' = A_1'^{,m_{12}} = A_2'^{,m_{12}}$. Hence we can see that (P_∞) holds.

Assume $m_{12} \in 2\mathbb{N}+1$. Let $Z := (XY)^{(m_{12}-1)/2}X$. Then $Z := Y(XY)^{(m_{12}-1)/2}$. Since $Z(A') \cap A' \neq \emptyset$, we have $TZ \neq E$. Since $(TZ)^2 = (XY)^{m_{12}} = E$ and $\det(TZ) = 1$, we have $TZ = -E$. Then $Z(A') = -A'$. Hence, similarly as above, we can see that (P_∞) holds.

Step 2. (P_q) and $(Q_q) \Rightarrow (P_{q+1})$.

Let $w \in W$ be such that $\ell(w) = q + 1$. Let $i \in I$. Let $j \in I$ be such that $\ell(s_j w) < \ell(w)$. Let $w' := s_j w$. Then $\ell(w') = q$. By (3.2), $\gamma(w')(A) \subset A_j$. Hence $\gamma(w)(A) \subset \gamma(s_j)(A_j)$. So if $i \neq j$, (P_{q+1}) for this w holds.

Assume $j \neq i$. Let $w_{ij} \in W_{ij}$ be as in (Q_q) for w' in place of w. Then $\gamma(w)(A) = \gamma(s_j w')(A) \subset \gamma(s_j w_{ij})(A_{ij})$. By Step 1, either $\gamma(w)(A) \subset A_i$ or $\gamma(w)(A) \subset \gamma(s_i)(A_i)$ holds. Assume $\gamma(w)(A) \subset \gamma(s_i)(A_i)$. Then $\gamma(s_j w_{ij})(A_{ij}) \subset \gamma(s_i)(A_i)$. By Step 1, $\ell_{ij}(s_i s_j w_{ij}) = \ell_{ij}(s_j w_{ij}) - 1$. Then

$$\begin{aligned} \ell(s_i w) &\leq \ell_{ij}(s_i s_j w_{ij}) + \ell(w_{ij}^{-1} w') \\ &= \ell_{ij}(s_j w_{ij}) - 1 + \ell(w') - \ell_{ij}(w_{ij}) \\ &\leq \ell(w') \\ &= q. \end{aligned} \tag{3.7}$$

Hence $\ell(s_i w) = q$, as desired.

Step 3. (P_q) *and* $(Q_{q-1}) \Rightarrow (Q_q)$.

Let i, $j \in I$ with $i \neq j$. Let $w \in W$ with $\ell(w) = q$. By (P_q), there exist x, $y \in J_{0,1}$ such that $\gamma(w)(A) \subset \gamma(s_i)^x(A_i) \cap \gamma(s_j)^y(A_j)$. If $\gamma(w)(A) \subset A_i \cap A_j$, then (Q_q) for this w holds by letting $w_{ij} := e$.

Assume $\gamma(w)(A) \subset \gamma(s_i)(A_i)$. Let $w' := s_i w$. By (P_q), $\ell(w') = q - 1$. By (Q_{q-1}), there exists $w'_{ij} \in W_{ij}$ such that $\gamma(w')(A) \subset \gamma(w'_{ij})(A_{ij})$ and $\ell(w') = \ell((w'_{ij})^{-1} w') + \ell_{ij}(w'_{ij})$. By Step 1, $\gamma(w)(A) \subset \gamma(s_i w'_{ij})(A_{ij}) \subset \gamma(s_i)(A_i)$, and $\ell_{ij}(w'_{ij}) = \ell_{ij}(s_i w'_{ij}) - 1$. Hence

$$\begin{aligned} \ell(w) &= \ell(w') + 1 \\ &= \ell((w'_{ij})^{-1} w') + \ell_{ij}(w'_{ij}) + 1 \\ &= \ell((w'_{ij})^{-1} w') + \ell_{ij}(s_i w'_{ij}) \\ &= \ell((s_i w'_{ij})^{-1} w) + \ell_{ij}(s_i w'_{ij}), \end{aligned} \tag{3.8}$$

as desired. This completes the proof. □

4 Dual of ρ

Let $\rho : W \to \mathrm{GL}(V)$ be as in Lemma 2.2. Define the group homomorphism $\rho^* : W \to \mathrm{GL}(V^*)$ by $(\rho^*(w)(f))(v) := f(\rho(w)^{-1}(v))$.

Lemma 4.1. $\ker \rho^* = \ker \rho$.

Let $i \in I$. Let $H_i := \{f \in V^* | f(v_i) = 0\}$. Then $\dim H_i = N - 1$. Let $A_i := \{f \in V^* | f(v_i) > 0\}$. Since $\rho(s_i)(v_i) = -v_i$, we have

Lemma 4.2. (1) $\rho^*(s_i)(A_i) = \{f \in V^* | f(v_i) < 0\}$.

(2) $\forall f \in H_i$, $\rho^*(s_i)(f) = f$.

(3) $V^* = A_i \cup H_i \cup \rho^*(s_i)(A_i)$ *(disjoint).*

Let $v_i^* \in V^*$ be such that $v_i^*(v_j) = \delta_{ij}$. Then $\{v_i^* | i \in I\}$ is a basis of V^*. Let $A := \cap_{i\in I} A_i$. Then $A \neq \emptyset$ since $\sum_{i\in I} v_i^* \in A$. For i, $j \in I$ with $i \neq j$, $H_i \neq H_j$ since $v_j^* \in H_i \setminus H_j$.

For i, $j \in I$ with $i \neq j$, let $A_{ij} := A_i \cap A_j$. Then $A_{ij} \neq \emptyset$ since $A \subset A_{ij}$.

Lemma 4.3. *Let* i, $j \in I$ *be such that* $i \neq j$.

(1) $(\rho^*)_{|W_{ij}}$ *is injective.*

(2) $\forall w \in W_{ij} \setminus \{e\}$, $\rho^*(w)(A_{ij}) \cap A_{ij} = \emptyset$.

Proof. Let $K_{ij} := \mathbb{R} v_i^* \oplus \mathbb{R} v_j^*$. Let $H_{ij} := H_i \cap H_j$. Then $\{v_k^* | k \in I \setminus \{i, j\}\}$ is a basis of H_{ij}. In particular, $\dim H_{ij} = N - 2$, and $V^* = K_{ij} \oplus H_{ij}$. Note that $\forall w \in W_{ij}$, $\forall f \in H_{ij}$, $\rho^*(w)(f) = f$. Let $A'_{ij} := \mathbb{R}_{>0} v_i^* \oplus \mathbb{R}_{>0} v_j^*$. Then $A_{ij} = A'_{ij} \oplus H_{ij}$.

Assume $m_{ij} < \infty$. Let $c_1 := \cos(\pi/m_{ij})$, and $c_2 := \sin(\pi/m_{ij})$. Since $m_{ij} \geq 2$, we have $c_2 > 0$. Define the linear isomorphism $P : K_{ij} \to \mathbb{R}^2$ by $P(v_i^*) := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $P(v_j^*) := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. We have

$$
\begin{aligned}
P\rho^*(s_i)P^{-1} &= \begin{bmatrix} 1 & c_1 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2c_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -c_1/c_2 \\ 0 & 1/c_2 \end{bmatrix} \\
&= \begin{bmatrix} 2c_1^2 - 1 & 2c_1c_2 \\ 2c_1c_2 & 1 - 2c_1^2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix},
\end{aligned} \tag{4.1}
$$

and

$$
P\rho^*(s_j)P^{-1} = \begin{bmatrix} 1 & c_1 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -1 & 2c_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -c_1/c_2 \\ 0 & 1/c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.2}
$$

Note that $P\rho^*(s_is_j)P^{-1} = \begin{bmatrix} \cos(2\pi/m_{ij}) & -\sin(2\pi/m_{ij}) \\ \sin(2\pi/m_{ij}) & \cos(2\pi/m_{ij}) \end{bmatrix}$. Since $P(A'_{ij}) = \mathbb{R}_{>0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \mathbb{R}_{>0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we can see that this lemma for $m_{ij} < \infty$ holds.

Assume $m_{ij} = \infty$. Define the linear isomorphism $Q : K_{ij} \to \mathbb{R}^2$ by $Q(v_i^*) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $Q(v_j^*) := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have

$$
Q\rho^*(s_i)Q^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.3}
$$

and

$$
Q\rho^*(s_j)Q^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{4.4}
$$

Note that $Q\rho^*((s_js_i)^k)Q^{-1} = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$ for $k \in \mathbb{Z}$. Hence for $k \in \mathbb{Z}$, we have $Q(\rho^*((s_js_i)^k)(A'_{ij})) = \mathbb{R}_{>0} \begin{bmatrix} 2k+1 \\ 1 \end{bmatrix} \oplus \mathbb{R}_{>0} \begin{bmatrix} 2k \\ 1 \end{bmatrix}$, and $Q(\rho^*((s_js_i)^k s_i)(A'_{ij})) = \mathbb{R}_{>0} \begin{bmatrix} 2k-1 \\ 1 \end{bmatrix} \oplus \mathbb{R}_{>0} \begin{bmatrix} 2k \\ 1 \end{bmatrix}$. Then this lemma for $m_{ij} = \infty$ holds. This completes the proof. □

By the above lemmas, and Theorem 3.2, we have

Theorem 4.4. (1) $\forall w \in W \setminus \{e\}$, $\rho^*(w)(A) \cap A = \emptyset$. *In particular,* ρ^* *and* ρ *are injective.*

(2) *Let* $i \in I$. *Let* $w \in W$. *Then either* $\rho^*(w)(A) \subset A_i$ *or* $\rho^*(w)(A) \subset \rho^*(s_i)(A_i)$ *holds. Moreover,* $\rho^*(w)(A) \subset A_i \Leftrightarrow \ell(s_iw) = \ell(w) + 1$.

(3) *Let* $i, j \in I$ *be such that* $i \neq j$. *Let* $w \in W$. *Then* $\forall w \in W$, $\exists w_{ij} \in W_{ij}$, $\rho^*(w)(A) \subset \rho^*(w_{ij})(A_{ij})$, $\ell(w) = \ell(w_{ij}^{-1}w) + \ell(w_{ij})$.

5 Root system

Let

$$\Phi := \{\, \rho(w)(v_i) \in V \,|\, w \in W,\, i \in I \,\}. \tag{5.1}$$

Let

$$\Phi^+ := \Phi \cap (\oplus_{i\in I}\mathbb{R}_{\geq 0}v_i),\ \ \Phi^- := \Phi \cap (\oplus_{i\in I}\mathbb{R}_{\geq 0}(-v_i)). \tag{5.2}$$

Theorem 5.1. (1) $0 \notin \Phi$, $\Phi^+ \cap \Phi^- = \emptyset$, $\Phi^- = -\Phi^+$, *Moreover,* $\forall v \in \Phi$, $\mathbb{R}v \cap \Phi = \{v, -v\}$.

(2) $\Phi = \Phi^+ \cup \Phi^-$.

(3) *Let* $i \in I$, *and let* $w \in W$. *Then*

$$\rho(w)(v_i) \in \Phi^+ \ \Leftrightarrow\ \ell(ws_i) = \ell(w) + 1. \tag{5.3}$$

Proof. (1) Let $v \in \Phi^+$. Then $\exists w \in W$, $\exists i \in I$, $v = \rho(w)(v_i)$. Then $-v = \rho(ws_i)(v_i) \in \Phi$. Hence $-\Phi^+ \subset \Phi$. Clearly we have $-\Phi^+ \subset \Phi^-$. Similarly $-\Phi^- \subset \Phi^+$. Hence $\Phi^- = -\Phi^+$. The rest claims follow from the fact that $\forall v \in \Phi$, $(v, v) = 1$.

(2) Let A_i and A be as in Section 4. Note that $A = \oplus_{i\in I}\mathbb{R}_{>0}v_i^*$. Then we see

$$(\oplus_{i\in I}\mathbb{R}_{\geq 0}v_i) \setminus \{0\} = \{\, v \in V \,|\, \forall f \in A,\, f(v) > 0 \,\}. \tag{5.4}$$

Hence by (1), we have

$$\Phi^+ = \{\, v \in \Phi \,|\, \forall f \in A,\, f(v) > 0 \,\}. \tag{5.5}$$

Similarly we have $\Phi^- = \{\, v \in \Phi \,|\, \forall f \in A,\, f(v) < 0 \,\}$.

Let $v \in \Phi$. Then $\exists w \in W$, $\exists i \in I$, $v = \rho(w)(v_i)$. By Theorem 4.4 (2), $\rho^*(w^{-1})(A) \subset A_i$ or $\rho^*(w^{-1})(A) \subset \rho^*(s_i)(A_i)$. Assume $\rho^*(w^{-1})(A) \subset A_i$. Then $\forall f \in A$, $f(v) = \rho^*(w^{-1})(f)(v_i) > 0$. Hence $v \in \Phi^+$. Similarly we see that if $\rho^*(w^{-1})(A) \subset \rho^*(s_i)(A_i)$, then $v \in \Phi^-$. Hence $\Phi = \Phi^+ \cup \Phi^-$, as desired.

(3) Assume $\rho(w)(v_i) \in \Phi^+$. By (5.5), $\forall f \in A$, $\rho^*(w^{-1})(f)(v_i) = f(\rho(w)(v_i)) > 0$. Hence $\rho^*(w^{-1})(A) \subset A_i$. By Theorem 4.4 (2), we have $\ell(ws_i) = \ell(w) + 1$. Similarly, $\rho(w)(v_i) \in \Phi^- \Rightarrow \ell(ws_i) = \ell(w) - 1$. □

By Theorem 5.1 (1),

$$\forall i \in I,\ \{\, v \in \Phi^+ \,|\, \rho(s_i)(v) \in \Phi^- \,\} = \{v_i\}. \tag{5.6}$$

Lemma 5.2. *Let $w \in W$ and $i \in I$ be such that $\ell(s_i w) = \ell(w) + 1$. Let $j \in I$ and $k \in I \setminus \{i\}$ be such that $\rho(w)(v_j) \in \mathbb{R}_{>0}v_i \oplus \mathbb{R}_{>0}v_k$. Then $\ell(s_k w) = \ell(w) - 1$.*

Proof. Let $u_1 := \rho(w^{-1})(v_i)$. By (5.3), $u_1 \in \Phi^+$. Let $u_2 := \rho(w^{-1})(v_k) \in \Phi$. Let $x, y \in \mathbb{R}_{>0}$ be such that $\rho(w)(v_j) = xv_i + yv_k$. Then $xu_1 + yu_2 = v_j$. Hence $u_2 \in \Phi^-$. By (5.3), $\ell(s_k w) = \ell(w) - 1$, as desired. □

Let $i, j \in I$ with $i \neq j$. Let $C_{i,j;0} := e$. For $n \in \mathbb{N}$. let $C_{i,j;n} := s_i C_{j,i;n-1}$. If $n \in J_{0,m_{ij}}$, $\ell(C_{i,j;n}) = n$.

Assume $m_{ij} < \infty$. Then $C_{i,j;m_{ij}} = C_{j,i;m_{ij}}$. Let $C_{ij} := C_{i,j;m_{ij}-1}$. Let

$$o_{ij} := \begin{cases} j & \text{if } m_{ij} \in 2\mathbb{N}, \\ i & \text{if } m_{ij} \in 2\mathbb{N}+1. \end{cases} \tag{5.7}$$

Then

$$s_j C_{ij} = C_{ij} s_{o_{ij}}. \tag{5.8}$$

By (5.3), $\rho(C_{ij})(v_{o_{ij}}) \in \Phi^+$. By (5.8), $\rho(s_j C_{ij})(v_{o_{ij}}) = -\rho(C_{ij})(v_{o_{ij}})$. By (5.6), we have

$$\rho(C_{ij})(v_{o_{ij}}) = v_j. \tag{5.9}$$

Lemma 5.3. *Let $w \in W \setminus \{e\}$. Let $i \in I$ be such that $\ell(s_i w) = \ell(w) - 1$. Assume that $\exists k, \exists j \in I$, $\rho(w)(v_k) = v_j$. Then $i \neq j$, $m_{ij} - 1 \leq \ell(w)$, $\ell(C_{ij}^{-1} w) = \ell(w) - m_{ij} + 1$. (In particular, $m_{ij} < \infty$.) Moreover, $\rho(C_{ij}^{-1} w)(v_k) = v_{o_{ij}}$.*

Proof. If $\ell(w) = 1$, this lemma is clear since $i \neq j = k$ and $m_{ij} = 2$.

Assume $\ell(w) \geq 2$. By (5.3), we have $\ell(ws_k) = \ell(w) + 1$, so $\ell(s_i w s_k) = \ell(w)$. Assume $i = j$. Then $\rho(s_i w)(v_k) = -v_i \in \Phi^-$, which is contradicition by (5.3). Hence $i \neq j$. If $m_{ij} = 2$, this lemma for m_{ij} holds since $C_{ij} = s_i$. Assume $m_{ij} \geq 3$. Assume that $\exists n \in J_{1,\ell(w)}$ and $\ell(C_{i,j;n}^{-1} w) = \ell(w) - n$. Assume that if $m_{ij} < \infty$, $n \leq m_{ij} - 2$. Then $\rho(C_{i,j;n}^{-1} w)(v_k) = \rho(C_{i,j;n}^{-1})(v_j)$. Since $\ell(C_{i,j;n}^{-1} s_j) = n + 1$, by (5.3), $\rho(C_{i,j;n}^{-1})(v_j) \in (\mathbb{R}_{\geq 0}v_i \oplus \mathbb{R}_{\geq 0}v_j) \cap \Phi^+$. Let $x \in \{i, j\}$. We easily see $\ell(s_x C_{i,j;n}^{-1} s_j) = \ell(s_x C_{i,j;n}^{-1}) + 1$, By (5.3), $\rho(s_x C_{i,j;n}^{-1})(v_j) \in (\mathbb{R}_{\geq 0}v_i \oplus \mathbb{R}_{\geq 0}v_j) \cap \Phi^+$. By (5.6), $\rho(C_{i,j;n}^{-1})(v_j) \neq v_x$. Hence $n \leq \ell(w) - 1$. By Theorem 5.1 (1), $\rho(C_{i,j;n}^{-1})(v_j) \in (\mathbb{R}_{>0}v_i \oplus \mathbb{R}_{>0}v_j) \cap \Phi^+$. By Lemma 5.2, we see that $\ell(C_{i,j;n+1}^{-1} w) = \ell(w) - n - 1$. By this argument, we see that this theorem holds. □

Using Theorem 4.4, for $w \in W$, we also see

$$\rho(w)(v_i) = v_j \ \Rightarrow \ ws_i = s_j w \tag{5.10}$$

since $\rho(w s_i w^{-1}) = \rho(s_j)$.

6 Matsumoto's theorem

Let $\widetilde{W}$ be the monoid defined by the generators $\tilde{s}_i$ $(i \in I)$ and the relations

$$(6.1)\qquad \underbrace{\tilde{s}_i\tilde{s}_j\tilde{s}_i\cdots}_{m_{ij}} = \underbrace{\tilde{s}_j\tilde{s}_i\tilde{s}_j\cdots}_{m_{ij}} \quad (i \neq j,\ m_{ij} < \infty).$$

Let $\tilde{e}$ denote the unit of $\widetilde{W}$.

Define the map $\tilde{\ell} : \widetilde{W} \to \mathbb{Z}_{\geq 0}$ by

$$(6.2)\qquad \tilde{\ell}(\tilde{w}) := \begin{cases} 0 & \text{if } \tilde{w} = \tilde{e}, \\ \min\{n \in \mathbb{N} \,|\, \exists i_x \in I\,(x \in J_{1,n}),\ \tilde{w} = \tilde{s}_{i_1}\cdots\tilde{s}_{i_n}\} & \text{otherwise.} \end{cases}$$

Then $\forall i \in I$, $\tilde{\ell}(\tilde{s}_i) = 1$. Moreover, $\forall \tilde{w}_1, \forall \tilde{w}_1 \in \widetilde{W}$, $\tilde{\ell}(\tilde{w}_1\tilde{w}_2) = \tilde{\ell}(\tilde{w}_1) + \tilde{\ell}(\tilde{w}_2)$. For $n \in \mathbb{Z}_{\geq 0}$, let $\widetilde{W}^{(n)} := \{\, \tilde{w} \in \widetilde{W} \,|\, \tilde{\ell}(\tilde{w}) = n \,\}$.

Define the monoid homomorphism $\tilde{\iota} : \widetilde{W} \to W$ by $\tilde{\iota}(\tilde{s}_i) := s_i$ $(i \in I)$.

Let $i, j \in I$ with $i \neq j$. Let $\tilde{C}_{i,j;0} := \tilde{e}$. For $n \in \mathbb{N}$. let $\tilde{C}_{i,j;n} := \tilde{s}_i\tilde{C}_{j,i;n-1}$.

Assume $m_{ij} < \infty$. Let $\tilde{C}_{ij} := \tilde{C}_{i,j;m_{ij}-1}$. Then

$$(6.3)\qquad \tilde{s}_j\tilde{C}_{ij} = \tilde{C}_{ij}\tilde{s}_{o_{ij}}.$$

Theorem 6.1. *Let $w \in W$. Then*

$$(6.4)\qquad |\tilde{\iota}^{-1}(\{w\}) \cap \widetilde{W}^{(\ell(w))}| = 1.$$

Proof. Let $q := \ell(W)$. If $q = 0$, (6.4) is clear. If $q = 1$, (6.4) is clear from (5.6).

Assume that $q \geq 2$. Let $\tilde{w}, \tilde{z} \in \widetilde{W}^{(q)}$ be such that $\tilde{\iota}(\tilde{w}) = \tilde{\iota}(\tilde{z}) = w$. Let us show

$$(6.5)\qquad \tilde{w} = \tilde{z}.$$

Let $j \in I$ and $\tilde{z}' \in \widetilde{W}^{(q-1)}$ be such that $\tilde{z} = \tilde{z}'\tilde{s}_j$. Note $\ell(\tilde{\iota}(\tilde{z}')) = q - 1$. Since $\ell(ws_j) = q - 1$, by (5.3), we have $\rho(w)(v_j) \in \Phi^-$. Since $v_j \in \Phi^+$, it follows that $\exists \tilde{w}_1, \exists \tilde{w}_2 \in \widetilde{W}, \exists k \in I$, $\tilde{w} = \tilde{w}_1\tilde{s}_k\tilde{w}_2$, $\rho(\tilde{\iota}(\tilde{w}_2))(v_j) \in \Phi^+$, $\rho(\tilde{\iota}(\tilde{s}_k\tilde{w}_2))(v_j) \in \Phi^-$, where note $\ell(\iota(\tilde{w}_x)) = \tilde{\ell}(\tilde{w}_x)$ $(x \in J_{1,2})$. By (5.6), we have

$$(6.6)\qquad \rho(\tilde{\iota}(\tilde{w}_2))(v_j) = v_k.$$

By (5.10),

$$(6.7)\qquad \tilde{\iota}(\tilde{w}_2\tilde{s}_j) = \tilde{\iota}(\tilde{s}_k\tilde{w}_2),\ \ell(\tilde{\iota}(\tilde{s}_k\tilde{w}_2)) = \tilde{\ell}(\tilde{w}_2) + 1.$$

Assume $\tilde{\ell}(\tilde{w}_2) \in J_{0,q-2}$. By (6.7) and induction, we have $\tilde{w}_2\tilde{s}_j = \tilde{s}_k\tilde{w}_2$. Note that $\tilde{\iota}(\tilde{z}') = \tilde{\iota}(\tilde{z}\tilde{s}_j) = \tilde{\iota}(\tilde{w}\tilde{s}_j) = \tilde{\iota}(\tilde{w}_1\tilde{s}_k\tilde{w}_2\tilde{s}_j) = \tilde{\iota}(\tilde{w}_1\tilde{s}_k^2\tilde{w}_2) = \tilde{\iota}(\tilde{w}_1\tilde{w}_2)$. Since

$\ell(\tilde{\iota}(\tilde{z}')) = q-1$, by induction, we have $\tilde{z}' = \tilde{w}_1\tilde{w}_2$. Then we have (6.5) since $\tilde{w} = \tilde{w}_1\tilde{s}_k\tilde{w}_2 = \tilde{w}_1\tilde{w}_2\tilde{s}_j = \tilde{z}'\tilde{s}_j = \tilde{z}$.

Assume $\tilde{\ell}(\tilde{w}_2) = q-1$. Then $\tilde{w}_1 = \tilde{e}$. Let $i \in I$ and $\tilde{w}_2' \in \widetilde{W}$ be such that $\tilde{w}_2 = \tilde{s}_i\tilde{w}_2'$. By (6.6), Lemma 5.3 and induction, it follows that $m_{ik} \leq q$, and $\exists \tilde{w}_2'' \in \widetilde{W}$, $\tilde{w}_2 = \tilde{C}_{ik}\tilde{w}_2''$. Moreover, $\rho(\iota(\tilde{w}_2''))(v_j) = v_{o_{ik}}$. Since $m_{ik} \geq 2$, by (6.6) and induction, we have $\tilde{w}_2''\tilde{s}_j = \tilde{s}_{o_{ik}}\tilde{w}_2''$. By (6.3), we have $\tilde{C}_{ik}\tilde{s}_{o_{ik}} = \tilde{s}_k\tilde{C}_{ik}$. Hence $\tilde{w}_2\tilde{s}_j = \tilde{s}_k\tilde{w}_2 = \tilde{w}$. By induction, $\tilde{w}_2 = \tilde{z}'$. Hence we have (6.5), as desired. This completes the proof. □

Theorem 6.2. *Let $\tilde{w} \in \widetilde{W}$. Assume $\ell(\tilde{\iota}(\tilde{w})) < \tilde{\ell}(\tilde{w})$. Then*

$$\exists k \in I,\ \exists \tilde{w}_1,\ \exists \tilde{w}_2 \in \widetilde{W},\ \tilde{w} = \tilde{w}_1\tilde{s}_k^2\tilde{w}_2. \tag{6.8}$$

Proof. Let $q := \tilde{\ell}(\tilde{w})$. Since $\tilde{\ell}(\tilde{w}) - \ell(\tilde{\iota}(\tilde{w})) \in 2\mathbb{N}$, $q \geq 2$.

Assume $q = 2$. Let $i, j \in I$ be such that $i \neq j$. Then $\ell(s_is_j) = 2$ since $\rho(s_is_j)(v_j) = -v_j + (v_i, v_j)v_i \neq v_j$. Hence the claim holds.

Assume $q \geq 3$. Let $i \in I$ and $\tilde{w}' \in \widetilde{W}$ be such that $\tilde{w} = \tilde{w}'\tilde{s}_i$. If $\ell(\iota(\tilde{w}')) \leq q-3$, by inducition we see that the claim holds. Assume $\ell(\iota(\tilde{w}')) = q-1$ Then $\ell(\iota(\tilde{w})) = q-2$. Let $\tilde{z} \in \widetilde{W}^{(q-2)}$ be such that $\iota(\tilde{w}) = \iota(\tilde{z})$. Then $\iota(\tilde{z}\tilde{s}_i) = \iota(\tilde{w}')$. By Theorem 6.1, $\tilde{z}\tilde{s}_i = \tilde{w}'$. Hence $\tilde{w} = \tilde{z}\tilde{s}_i^2$. This completes the proof. □.

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