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Examples of the defining relations of the quantum affine superalgebras

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1 A Gröbner basis of a noncomutative homogeneous algebra

1.1 The semigroup of the sequences of integers

Let $\mathbb{K}$ be a field. Let $\mathbb{N}$ be the set of the positive integers. Let $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Throughout Sec. 1, we let $n \in \mathbb{N}$ be a fixed positive integer. Let $I := \{i \in \mathbb{N} | 1 \leq i \leq n\}$. Let $\mathcal{V} := \mathbb{Z}_+^n$, i.e., $\mathcal{V}$ is the direct product of n-copies of $\mathbb{Z}_+$.

Let $\mathcal{Z} := \cup_{i=0}^{\infty} I^i$ (disjoint). Here I^i is the direct product of i-copies of N for $i \geq 1$, and we agree that I^0 is the set formed by an only element ϕ, i.e., $I^0 = \{\phi\}$. For $\alpha = (\alpha_1, \ldots, \alpha_i) \in I^i$ and $\beta = (\beta_1, \ldots, \beta_j) \in I^j$, we set $\alpha\beta := (\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_j) \in I^{i+j}$; we agree that $\alpha\phi = \phi\alpha = \alpha$. We write $\alpha \preceq \beta$ if $\gamma\alpha\delta = \beta$ for some γ, $\delta \in \mathcal{Z}$. We write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$. For $\alpha \in I^i$ and $j \in I$, we denote by $w(\alpha)_j$ the number of the integers k such that $1 \leq k \leq i$ and $\alpha_k = j$; we agree that $w(\phi)_j = 0$. Define the map $w : \mathcal{Z} \to \mathcal{V}$ by $w(\alpha) := (w(\alpha)_1, \ldots, w(\alpha)_n)$. Let $|w(\alpha)| := w(\alpha)_1 + \cdots + w(\alpha)_n$. For $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{V}$, let $\mathcal{Z}_\mu := w^{-1}(\mu)$. For α, $\beta \in \mathcal{Z}$, we write $\alpha < \beta$ if one of the following holds.

(1) $|w(\alpha)| < |w(\beta)|$,

(2) $|w(\alpha)| = |w(\beta)|$, and there exists a $k \in \{1, \dots, n\}$ such that $w(\alpha)_u = w(\beta)_u$ for $u < k$ and $w(\alpha)_k < w(\beta)_k$,

(3) $w(\alpha) = w(\beta)$, and there exists a $k \in \{1, \dots, |w(\alpha)|\}$ such that $\alpha_u = \beta_u$ for $u < k$, and $\alpha_k < \beta_k$,

For a subset $\mathcal{S}$ of $\mathcal{Z}$, denote by $\min(\mathcal{S})$ the minimal element of $\mathcal{S}$ with respect to $\leq$.

Let $\alpha \in I^i$ and $\beta \in I^j$. If there exists $k \in \mathbb{N}$ such that $1 \leq k < \min\{i, j\}$ and $\alpha_{i-k+u} = \beta_u$ for every $1 \leq u \leq k$, we let $\alpha \vee_k \beta := (\alpha_1, \dots, \alpha_i, \beta_{k+1}, \dots, \beta_j) \in I^{i+j-k}$, $\alpha \setminus_k \beta := (\alpha_1, \dots, \alpha_{i-k}) \in I^{i-k}$ and $\alpha /_k \beta := (\beta_{k+1}, \dots, \beta_j) \in I^{j-k}$.

1.2 The free algebra generated by n-integers and its homogeneous ideal

Let $\mathcal{F}$ be a free $\mathbb{K}$-algebra with n-generators $x_1, \dots, x_n$. For $\alpha \in I^i$, let $x_\alpha := x_{\alpha_1} \cdots x_{\alpha_i}$; we agree that $x_\phi = 1$. Then $\{x_\alpha | \alpha \in \mathcal{Z}\}$ is a $\mathbb{K}$-basis of $\mathcal{F}$. For $\mu \in \mathcal{V}$, let $\mathcal{F}_\mu := \oplus_{\alpha \in \mathcal{Z}_\mu} \mathbb{K} x_\alpha$. For $f = \sum_{\alpha \in \mathcal{Z}_\mu} a_\alpha x_\alpha \in \mathcal{F}_\mu \setminus \{0\}$, there exists a unique $\gamma \in \mathcal{Z}_\mu$ such that $a_\gamma \neq 0$ and $a_\beta = 0$ for every $\beta > \gamma$; we denote the γ by $t(f)$, and let $T(f) := a_{t(f)}^{-1} f$. We also let $T(0) := 0$.

An element of $\oplus_{\mu \in \mathcal{V}} \mathcal{F}_\mu \setminus \{0\}$ is call *homogeneous*. Denote by $\mathcal{F}'$ the set of the homogeneous elements f of $\mathcal{F}$ with $T(f) = f$. Let $\mathcal{G}$ be a subset of $\mathcal{F}'$. If the map from $\mathcal{G}$ to $\mathcal{Z}$ defined by sending $f \in \mathcal{G}$ to $t(f) \in \mathcal{Z}$ is injective, then we call a subset $\mathcal{G}$ of $\mathcal{F}'$ *admissible* and denote by f_α for the element $f \in \mathcal{G}$ with $t(f) = \alpha$; let $t(\mathcal{G}) := \{t(f) \in \mathcal{Z} | f \in \mathcal{G}\}$.

Let $\mathcal{G} = \{f_\alpha | \alpha \in t(\mathcal{G})\}$ be an admissible subset. Let $f \in \mathcal{F}' \cup \{0\}$. Let $A_f := \{\alpha \in t(\mathcal{G}) | t(f) \succeq \alpha\}$ if $f \neq 0$; if $f = 0$, let A_f be an empty set. If A_f is empty, let $\Phi'_{\mathcal{G}}(f) := f$. If A_f is not empty, let $\Phi'_{\mathcal{G}}(f) := T(T(f) - x_\alpha f_\beta x_\gamma)$, where $\beta = \min(A_f)$, and $\alpha, \gamma \in \mathcal{Z}$ are such that $t(f) = \alpha\beta\gamma$. Clearly there exists an $i \in \mathbb{N}$ such that $(\Phi'_{\mathcal{G}})^{i+1}(f) = (\Phi'_{\mathcal{G}})^i(f)$; we denote it by $\Phi_{\mathcal{G}}(f)$. Let $B := \{\alpha \in t(\mathcal{G}) | \alpha \prec \beta \text{ for some } \beta \in t(\mathcal{G})\}$. If B is empty, let $\Xi'(\mathcal{G}) = \mathcal{G}$. If B is not empty, let $\beta = \min(B)$ and

$\Xi'(\mathcal{G}) = ((\mathcal{G} \setminus \{f_\beta\}) \cup \{\Phi_\mathcal{G}(f_\beta)\}) \setminus \{0\}\}$. Let $\Xi(\mathcal{G}) := (\Xi')^\infty(\mathcal{G})$. If $\mathcal{G}$ is a finite set, $\Xi(\mathcal{G}) = (\Xi')^i(\mathcal{G})$ for some $i \in \mathbb{N}$.

We say that a $\mathbb{K}$-vector subspace $\mathcal{I}$ of $\mathcal{F}$ is a *homogeneous ideal* if $\mathcal{I} = \oplus_{\mu\in\mathcal{V}}(\mathcal{I} \cap \mathcal{F}_\mu)$ and if $gf \in \mathcal{I}$ and $fg \in \mathcal{I}$ for every $g \in \mathcal{F}$ and every $f \in \mathcal{I}$.

Let $\mathcal{I}$ be a homogeneous ideal of $\mathcal{F}$. Let $\mu \in \mathcal{Z}$. Let $\mathcal{I}_\mu := \mathcal{I} \cap \mathcal{F}_\mu$. Let $C(\mathcal{I}, \mu) := \{t(f) \in \mathcal{Z}_\mu | f \in \mathcal{I}_\mu \setminus \{0\}\}$ and $B(\mathcal{I}, \mu) := \mathcal{Z}_\mu \setminus C(\mathcal{I}, \mu)$. For each $\alpha \in C(\mathcal{I}, \mu)$, we choose a $g_\alpha \in \mathcal{I}_\mu \setminus \{0\}$ such that $t(g_\alpha) = \alpha$ and $T(g_\alpha) = g_\alpha$. Then $\{g_\alpha | \alpha \in C(\mathcal{I}, \mu)\}$ is a $\mathbb{K}$-basis of $\mathcal{I}_\mu$, and $\{g_\alpha | \alpha \in C(\mathcal{I}, \mu)\} \cup \{x_\beta | \beta \in B(\mathcal{I}, \mu)\}$ is a basis of $\mathcal{F}_\mu$.

Let $\mathcal{I}$ be a homogeneous ideal of $\mathcal{F}$. We say that an admissible subset $\mathcal{G} = \{f_\alpha | \alpha \in t(\mathcal{G})\}$ of $\mathcal{I} \setminus \{0\}$ is a *Gröbner basis* if:

(i) $\mathcal{I} = \sum_{\alpha,\gamma\in\mathcal{Z},\, \beta\in t(\mathcal{G})} \mathbb{K} x_\alpha f_\beta x_\gamma$

(ii) $\Xi(\mathcal{G}) = \mathcal{G}$

(iii) For every $(\alpha, \beta, k) \in t(\mathcal{G})^2 \times \mathbb{N}$ such that $\alpha \vee_k \beta$ can be defined, we have

$$\Phi_\mathcal{G}(f_\alpha x_{\alpha/_k\beta} - x_{\alpha\setminus_k\beta} f_\beta) = 0. \tag{1.1}$$

Theorem 1.1. *Let $\mathcal{F}$ be the free $\mathbb{K}$-algebra introduced above. Let $\mathcal{I}$ be a homogeneous ideal of $\mathcal{F}$. Then there exists a Gröbner basis $\mathcal{G} = \{f_\alpha | \alpha \in t(\mathcal{G})\}$ of $\mathcal{I}$.*

We can prove the theorem easily.

Lemma 1.1. *Keep the notation in the above theorem. Let $\mu \in \mathcal{V}$ be such that $\mathcal{I}_\mu \neq \{0\}$.*

(1) For every $\alpha \in C(\mathcal{I}, \mu)$, there exists a $g_\alpha \in \mathcal{I}_\mu$ such that $g_\alpha = x_\gamma f_\theta x_\delta$ for some $\theta \in t(\mathcal{G})$ and some γ, $\delta \in \mathcal{Z}$.

(2) For each $\alpha \in C(\mathcal{I}, \mu)$, choose $g_\alpha \in \mathcal{I}_\mu$ satisfying the property of (1). Then $\{g_\alpha | \alpha \in C(\mathcal{I}, \mu)\}$ is a $\mathbb{K}$-basis of $\mathcal{I}_\mu$.

(3) $C(\mathcal{I}, \mu) = \{\alpha \in \mathcal{Z}_\mu | \alpha \succeq \gamma$ for some $\gamma \in t(\mathcal{G})\}$

Proof. Let $f'_\alpha := f_\alpha - x_\alpha$. Note that if $f'_\alpha \neq 0$, $t(f'_\alpha) < t(f_\alpha)$. We can prove the lemma by using (1.3) and the fact that

$$\begin{aligned}
&f_\alpha x_\beta x_\gamma - x_\alpha x_\beta f_\gamma \\
&\quad = f'_\alpha x_\beta x_\gamma - x_\alpha x_\beta f'_\gamma \\
&\quad = f'_\alpha x_\beta (f_\gamma - f'_\gamma) - (f_\alpha - f'_\alpha) x_\beta f'_\gamma \\
&\quad = f'_\alpha x_\beta f_\gamma - f_\alpha x_\beta f'_\gamma
\end{aligned} \tag{1.2}$$

Q.E.D.

1.3 How to construct a Gröbner basis

For μ, $\nu \in \mathcal{V}$, we write $\mu \succ \nu$ if there exist α, $\beta \in \mathcal{Z}$ such that $w(\alpha) = \mu$, $w(\beta) = \nu$ and $\alpha \succ \beta$. Let $\mathcal{I}$ be a homogeneous ideal of $\mathcal{F}$. Let $\mu \in \mathcal{V}$. Let $\mathcal{I}_{\preceq\mu} := \mathcal{I} \cap (\oplus_{\nu\preceq\mu}\mathcal{F}_\nu)$.

We say that an admissible subset $\mathcal{G} = \{f_\alpha | \alpha \in t(\mathcal{G})\}$ of $\mathcal{I} \setminus \{0\}$ is a μ-restricted *Gröbner basis* if:

(i) $\mathcal{I} = \sum_{\alpha,\gamma\in\mathcal{Z},\,\beta\in t(\mathcal{G})} \mathbb{K} x_\alpha f_\beta x_\gamma$

(ii) $\Xi(\mathcal{G} \cap \mathcal{I}_{\preceq\mu}) = \mathcal{G} \cap \mathcal{I}_{\preceq\mu}$

(iii) For every $(\alpha, \beta, k) \in t(\mathcal{G})^2 \times \mathbb{N}$ such that $\alpha \vee_k \beta$ with $w(\alpha \vee_k \beta) \preceq \mu$ can be defined, we have

$$\Phi_{\mathcal{G}}(f_\alpha x_{\alpha/_k\beta} - x_{\alpha\backslash_k\beta} f_\beta) = 0. \tag{1.3}$$

Let $\mathcal{G}$ be an admissible subset of $\mathcal{I}\setminus\{0\}$. Assume that for every $\nu \prec \mu$, $\mathcal{G}$ is a a ν-restricted Gröbner basis of $\mathcal{I}$. Replace $\mathcal{G}$ by $(\mathcal{G} \setminus (\mathcal{G} \cap \mathcal{I}_{\preceq\mu})) \cup \Xi(\mathcal{G} \cap \mathcal{I}_{\preceq\mu})$. Let $S_1 := \{\alpha \in t(\mathcal{G}) | \exists\beta,\ \alpha \vee_k \beta$ with $w(\alpha \vee_k \beta) = \mu$ can be defined and $\Phi_{\mathcal{G}}(f_\alpha x_{\alpha/_k\beta} - x_{\alpha\backslash_k\beta} f_\beta) \neq 0\ \}$. If $S_1 = \emptyset$, let $\Theta'_\mu(\mathcal{G}) := \mathcal{G}$. Assume $S_1 \neq \emptyset$. Let $\alpha := \min(S_1)$. Let $S_2 := \{\beta \in t(\mathcal{G}) |\ \alpha \vee_k \beta$ with $w(\alpha \vee_k \beta) = \mu$ can be defined and $\Phi_{\mathcal{G}}(f_\alpha x_{\alpha/_k\beta} - x_{\alpha\backslash_k\beta} f_\beta) \neq 0\ \}$. Let $\beta := \min(S_2)$. Let $\gamma := \alpha \vee_k \beta$ and let $f_\gamma := \Phi_{\mathcal{G}}(f_\alpha x_{\alpha/_k\beta} - x_{\alpha\backslash_k\beta} f_\beta)$. Let $\Theta'_\mu(\mathcal{G}) := \mathcal{G} \cup \{f_\gamma\}$.

Let $\Theta'_\mu(\mathcal{G})$ be as above. Let $\Theta_\mu(\mathcal{G}) := (\Theta_\mu)^\infty(\mathcal{G})$. then $\Theta_\mu(\mathcal{G})$ is a μ-restricted Gröbner basis of $\mathcal{I}$.

By using Θ_μ for all μ, we have a Gröbner basis of $\mathcal{I}$.

By using a Gröbner basis, I checked that any of the defining relations of the finite-type Lie superalgebras in my paper [Y0] cannot be dropped. I have not yet checked the same for those of the affine Lie superalgebras in [Y1].

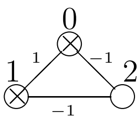
2 Defining relations

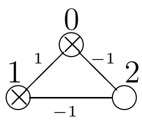
Notation:

$$(ad_{+,a}(X))(Y) := [X,Y]_{+,a} := XY + aYX,$$
$$(ad_{-,a}(X))(Y) := [X,Y]_{-,a} := XY - aYX.$$

For $m \geq 1$, $\Big((ad_{+,a}(X))\Big)^m(Y) := \Big((ad_{+,a}(X))\Big)^{m-1}((ad_{+,a}(X))(Y))$ and $\Big((ad_{-,a}(X))\Big)^m(Y) := \Big((ad_{-,a}(X))\Big)^{m-1}((ad_{-,a}(X))(Y))$.

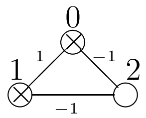
In this section, we always treat $\mathbb{C}(q^{\frac{1}{2}})$-algebras.

2.1 Defining relations of $U_q(\widehat{\mathrm{sl}}(1|2),$ )

Theorem 2.1. *Define the Cartan matrix $A = (a_{ij})$ and the parity $p(i)$ of $\widehat{\mathrm{sl}}(1|2) = \mathbf{A}^{(1)}(0,1)$, the affine Lie superalgebra whose Dynkin diagram is*  *, by the following.*

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$p(0) = p(1) = 1,\ p(2) = 0.$$

Then the defining relations of $U_q(\widehat{\mathrm{sl}}(1|2)) = U_q(\widehat{\mathrm{sl}}(1|2),$ 

$)$ *satisfied by the generators*

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, E_0, E_1, E_2, F_0, F_1, F_2,$$

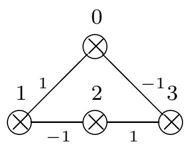
are the following.

$$\begin{cases} \sigma^2 = 1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma = K_i^{\pm\frac{1}{2}},\ \sigma E_i \sigma = (-1)^{p(i)} E_i,\ \sigma F_i \sigma = (-1)^{p(i)} F_i, \\ K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1,\ K_i^{\frac{1}{2}} K_j^{\frac{1}{2}} = K_j^{\frac{1}{2}} K_i^{\frac{1}{2}} \\ K_i^{\frac{1}{2}} E_j K_i^{-\frac{1}{2}} = q^{\frac{a_{ij}}{2}} E_j,\ K_i^{\frac{1}{2}} F_j K_i^{-\frac{1}{2}} = q^{-\frac{a_{ij}}{2}} F_j, \\ E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \end{cases}$$

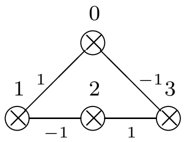
$$\begin{cases} E_1^2 = E_0^2 = 0,\ [E_2, [E_2, E_1]_{-,q}]_{-,q^{-1}} = 0,\ [E_2, [E_2, E_0]_{-,q}]_{-,q^{-1}} = 0, \\ [E_0, [E_1, [E_0, [E_1, E_2]_{-,q}]_{+,1}]_{-,1}]_{+,q^{-1}} = [E_1, [E_0, [E_1, [E_0, E_2]_{-,q}]_{+,1}]_{-,1}]_{+,q^{-1}} \end{cases}$$

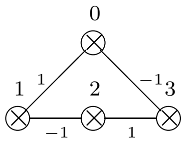
$$\begin{cases} F_1^2 = F_0^2 = 0,\ [F_2, [F_2, F_1]_{-,q}]_{-,q^{-1}} = 0,\ [F_2, [F_2, F_0]_{-,q}]_{-,q^{-1}} = 0, \\ [F_0, [F_1, [F_0, [F_1, F_2]_{-,q}]_{+,1}]_{-,1}]_{+,q^{-1}} = [F_1, [F_0, [F_1, [F_0, F_2]_{-,q}]_{+,1}]_{-,1}]_{+,q^{-1}} \end{cases}$$

□

2.2. Defining relations of $U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

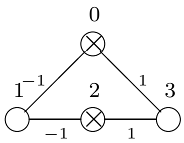
$),$

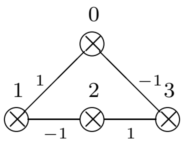
$U_q(\widehat{\mathrm{sl}}(2|2),$ 

$)$ and $U_q(\mathbf{A}^{(1)}(1,1),$ 

$)$

$\mathbf{D}^{(1)}(2,1;-1)$, $\widehat{\mathrm{sl}}(2|2)$, $\mathbf{A}^{(1)}(1,1)$ are Lie superalgebras whose Dynkin diagram is



or


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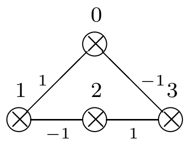
Notice that there is a canonical surjective map from $\mathbf{D}^{(1)}(2,1;-1)$ to $\widehat{\mathrm{sl}}(2|2)$, and there is a canonical surjective map from $\widehat{\mathrm{sl}}(2|2)$ to $\mathbf{A}^{(1)}(1,1)$. The $\mathbf{D}^{(1)}(2,1;-1)$ is the universal central extension of $\mathbf{A}^{(1)}(1,1)$ and it is also the universal central extension of $\widehat{\mathrm{sl}}(2|2)$ (See [IK]; see also [Y2]).

$$\mathbf{D}^{(1)}(2,1;-1) \twoheadrightarrow \widehat{\mathrm{sl}}(2|2) \twoheadrightarrow \mathbf{A}^{(1)}(1,1)$$

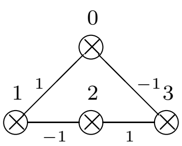
Theorem 2.2. *Define the Cartan matrix $A=(a_{ij})$ and the parity $p(i)$ of $\mathbf{A}^{(1)}(1,1)$, the affine Lie superalgebra whose Dynkin diagram is*



, by the following.

$$A=\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$$p(0):=p(1):=p(2):=p(3):=1.$$

Then the defining relations of $U_q(\mathbf{D}^{(1)}(2,1;-1)) = U_q(\mathbf{D}^{(1)}(2,1;-1),$


) satisfied by the generators

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, K_3^{\pm\frac{1}{2}}, E_0, E_1, E_2, E_3, F_0, F_1, F_2, F_3,$$

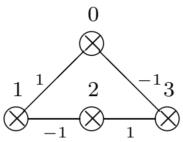
are the following.

$$\begin{cases} \sigma^2 = 1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma = K_i^{\pm\frac{1}{2}},\ \sigma E_i \sigma = (-1)^{p(i)}E_i,\ \sigma F_i \sigma = (-1)^{p(i)}F_i, \\ K_i^{\frac{1}{2}}K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}}K_i^{\frac{1}{2}} = 1,\ K_i^{\frac{1}{2}}K_j^{\frac{1}{2}} = K_j^{\frac{1}{2}}K_i^{\frac{1}{2}} \\ K_i^{\frac{1}{2}}E_jK_i^{-\frac{1}{2}} = q^{\frac{a_{ij}}{2}}E_j,\ K_i^{\frac{1}{2}}F_jK_i^{-\frac{1}{2}} = q^{-\frac{a_{ij}}{2}}F_j, \\ E_iF_j - (-1)^{p(i)p(j)}F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}} \end{cases}$$

$$\begin{cases} E_0^2 = E_1^2 = E_2^2 = E_3^2 = 0, \\ [E_2,[[E_1,E_2]_{+,q},E_3]_{-,q^{-1}}]_{+,1} = 0, \quad [E_3,[[E_2,E_3]_{+,q^{-1}},E_0]_{-,q}]_{+,1} = 0, \\ [E_0,[[E_3,E_0]_{+,q},E_1]_{-,q^{-1}}]_{+,1} = 0, \quad [E_1,[[E_0,E_1]_{+,q^{-1}},E_2]_{-,q}]_{+,1} = 0, \end{cases}$$

$$\begin{cases} F_0^2 = F_1^2 = F_2^2 = F_3^2 = 0, \\ [F_2,[[F_1,F_2]_{+,q},F_3]_{-,q^{-1}}]_{+,1} = 0, \quad [F_3,[[F_2,F_3]_{+,q^{-1}},F_0]_{-,q}]_{+,1} = 0, \\ [F_0,[[F_3,F_0]_{+,q},F_1]_{-,q^{-1}}]_{+,1} = 0, \quad [F_1,[[F_0,F_1]_{+,q^{-1}},F_2]_{-,q}]_{+,1} = 0, \end{cases}$$

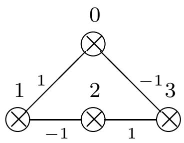
□

We have the following in $U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

$)$.

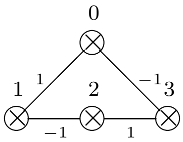
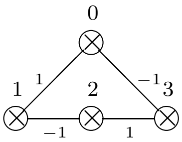
$$\begin{cases} [E_2,[E_1,E_3]_{+,1}]_{-,1} = 0, \quad [E_0,[E_1,E_3]_{+,1}]_{-,1} = 0, \\ [E_1,[E_0,E_2]_{+,1}]_{-,1} = 0, \quad [E_3,[E_0,E_2]_{+,1}]_{-,1} = 0, \end{cases}$$

$$\begin{cases} [F_2,[F_1,F_3]_{+,1}]_{-,1} = 0, \quad [F_0,[F_1,F_3]_{+,1}]_{-,1} = 0, \\ [F_1,[F_0,F_2]_{+,1}]_{-,1} = 0, \quad [F_3,[F_0,F_2]_{+,1}]_{-,1} = 0, \end{cases}$$

Hence $[E_1,E_3]_{+,1},\ [E_0,E_2]_{+,1},\ [F_1,F_3]_{+,1},\ [F_0,F_2]_{+,1}$ are elements in the center of $U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

$)$.

For $i = 1, 2, 3$, define the three automorphisms

$T_{\omega_i} : U_q(\mathbf{D}^{(1)}(2,1;-1),$  $) \to U_q(\mathbf{D}^{(1)}(2,1;-1),$  $)$
by

$$T_{\omega_i}(\sigma) = \sigma,$$

$$T_{\omega_i}(K_j) = K_j,\ T_{\omega_i}(E_j) = E_j,\ T_{\omega_i}(F_j) = F_j \text{ for } j \neq i \text{ and } j \neq 0,$$

$$T_{\omega_i}(K_i) = \frac{K_i}{K_0K_1K_2K_3},\ T_{\omega_i}(K_0) = K_iK_0K_1K_2K_3$$

$$T_{\omega_1}(E_1) = -[F_2,[F_3,F_0]_{+,q}]_{-,q^{-1}}\frac{K_0K_1K_2K_3}{K_1}$$

$$T_{\omega_1}(F_1) = \frac{K_1}{K_0K_1K_2K_3}[E_2,[E_3,E_0]_{+,q}]_{-,q^{-1}},$$

$$T_{\omega_1}(E_0) = -[[E_1,E_0]_{+,q^{-1}},[E_2,[E_3,E_0]_{+,q}]_{-,q^{-1}}]_{-,q},$$

$$T_{\omega_1}(F_0) = -[[F_1,F_0]_{+,q^{-1}},[F_2,[F_3,F_0]_{+,q}]_{-,q^{-1}}]_{-,q},$$

$$T_{\omega_2}(E_2) = -[[F_1,[F_3,F_0]_{+,q}]_{-,q^{-1}}\frac{K_0K_1K_2K_3}{K_2}$$

$$T_{\omega_2}(F_2) = \frac{K_2}{K_0K_1K_2K_3}[[E_1,[E_3,E_0]_{+,q}]_{-,q^{-1}},$$

$$T_{\omega_2}(E_0) = [[E_2,[E_1,E_0]_{+,q^{-1}}]_{-,q},[E_3,E_0]_{-,q}]_{-,q^{-1}},$$

$$T_{\omega_2}(F_0) = -[[F_2,[F_1,F_0]_{+,q^{-1}}]_{-,q},[F_3,F_0]_{-,q}]_{-,q^{-1}},$$

$$T_{\omega_3}(E_3) = -[F_2,[F_1,F_0]_{+,q^{-1}}]_{-,q}\frac{K_0K_1K_2K_3}{K_3}$$

$$T_{\omega_3}(F_3) = \frac{K_3}{K_0K_1K_2K_3}[E_2,[E_1,E_0]_{+,q^{-1}}]_{-,q},$$

$$T_{\omega_3}(E_0) = [[E_3,[E_2,[E_1,E_0]_{+,q^{-1}}]_{-,q}]_{+,1},E_0]_{-,1},$$

$$T_{\omega_3}(F_0) = [[F_3,[F_2,[F_1,F_0]_{+,q^{-1}}]_{-,q}]_{+,1},F_0]_{-,1}$$

The inverse $T_{\omega_i}^{-1}$ satisfies the following.

$$T_{\omega_i}^{-1}(K_i) = K_i(K_0K_1K_2K_3),\ T_{\omega_i}^{-1}(K_0) = K_1^{-1}K_2^{-1}K_3^{-1},$$

$$T_{\omega_1}^{-1}(E_1) = [[E_1, E_0]_{+,q^{-1}}, [[E_1, E_2]_{+,q}, E_3]_{-,q^{-1}}]_{-,q},$$
$$T_{\omega_1}^{-1}(F_1) = [[F_1, F_0]_{+,q^{-1}}, [[F_1, F_2]_{+,q}, F_3]_{-,q^{-1}}]_{-,q},$$
$$T_{\omega_2}^{-1}(E_2) = -[[[E_2, E_1]_{+,q}, E_0]_{-,q^{-1}}, [E_2, E_3]_{+,q^{-1}}]_{-,q},$$
$$T_{\omega_2}^{-1}(F_2) = -[[[F_2, F_1]_{+,q}, F_0]_{-,q^{-1}}, [F_2, F_3]_{+,q^{-1}}]_{-,q},$$
$$T_{\omega_3}^{-1}(E_3) = [E_3, [[[[E_3, E_2]_{+,q^{-1}}, E_1]_{-,q}, E_0]_{+,1}]_{-,1},$$
$$T_{\omega_3}^{-1}(F_3) = [F_3, [[[[F_3, F_2]_{+,q^{-1}}, F_1]_{-,q}, F_0]_{+,1}]_{-,1},$$
$$T_{\omega_1}^{-1}(E_0) = -K_1^{-1}K_2^{-1}K_3^{-1}[[F_1, F_2]_{+,q}, F_3]_{-,q^{-1}},$$
$$T_{\omega_1}^{-1}(F_0) = [[E_1, E_2]_{+,q}, E_3]_{-,q^{-1}}K_1K_2K_3,$$
$$T_{\omega_2}^{-1}(E_0) = -K_1^{-1}K_2^{-1}K_3^{-1}[[F_2, F_3]_{+,q^{-1}}, F_1]_{-,q},$$
$$T_{\omega_2}^{-1}(F_0) = -[[E_2, E_3]_{+,q^{-1}}, E_1]_{-,q}K_1K_2K_3,$$
$$T_{\omega_3}^{-1}(E_0) = K_1^{-1}K_2^{-1}K_3^{-1}[[F_3, F_2]_{+,q^{-1}}, F_1]_{-,q},$$
$$T_{\omega_3}^{-1}(F_0) = -[[E_3, E_2]_{+,q^{-1}}, E_1]_{-,q}K_1K_2K_3,$$

For $i \in \{1, 2, 3\}$, set

$$h_{i,+} := \frac{1}{(K_1K_1K_2K_3)^{\frac{1}{2}}}[E_i, T_{\omega_i}(K_i^{-1}F_i)]_{+,1},$$

$$h_{i,-} := (K_1K_1K_2K_3)^{\frac{1}{2}}[F_i, T_{\omega_i}(E_iK_i)]_{+,1},$$

For $m \in \mathbb{Z}$, we have the following.

$$[h_{2,\pm}, T_{\omega_1}^m(E_1)]_{-,1} = \frac{1}{(K_1K_1K_2K_3)^{\frac{1}{2}}}T_{\omega_1}^{m\mp1}(E_1),$$

$$[h_{2,\pm}, T_{\omega_1}^m(F_1)]_{-,1} = -(K_1K_1K_2K_3)^{\frac{1}{2}}T_{\omega_1}^{m\pm1}(F_1),$$

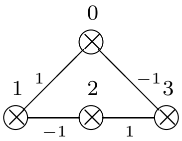
$$[h_{1,\pm}, T_{\omega_2}^m(E_2)]_{-,1} = -\frac{1}{(K_1K_1K_2K_3)^{\frac{1}{2}}}T_{\omega_2}^{m\mp1}(E_2),$$

$$[h_{1,\pm}, T^m_{\omega_2}(F_2)]_{-,1} = (K_1K_1K_2K_3)^{\frac{1}{2}} T^{m\pm 1}_{\omega_2}(F_2),$$

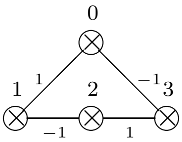
$$[h_{2,\pm}, T^m_{\omega_3}(E_3)]_{-,1} = -\frac{1}{(K_1K_1K_2K_3)^{\frac{1}{2}}} T^{m\mp 1}_{\omega_3}(E_3),$$

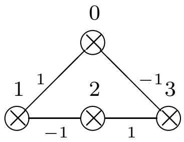
$$[h_{2,\pm}, T^m_{\omega_3}(F_3)]_{-,1} = (K_1K_1K_2K_3)^{\frac{1}{2}} T^{m\pm 1}_{\omega_3}(F_3).$$

Thus we can calculate $T^m_{\omega_i}(E_i)$ and $T^m_{\omega_i}(F_i)$ inductively. Then

$U_q(\widehat{\mathrm{sl}}(2|2),$ 

$)$ is the quotient algebra

$U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

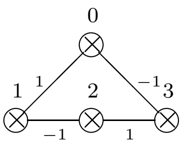
$)/I$ of $U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

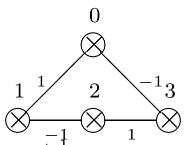
$),$
where I is the ideal generated by

$$[E_1, T^{-m}_{\omega_3}(E_3)]_{+,1} \ (m \geq 1), \ [F_1, T^{-m}_{\omega_3}(F_3)]_{+,1} \ (m \geq 1),$$

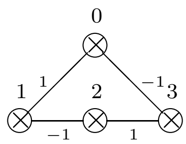
and

$$\Xi([E_1, T^{-m}_{\omega_3}(E_3)]_{+,1}) \ (m \geq 1), \ \Xi([F_1, T^{-m}_{\omega_3}(F_3)]_{+,1}) \ (m \geq 1),$$

where $\Xi : U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

$) \to U_q(\mathbf{D}^{(1)}(2,1;-1),$ 

$)$
is a $\mathbb{C}$-algebra automorphism such that $\Xi(q) = q^{-1}$, $\Xi(\sigma) = \sigma$, $\Xi(K_1^{\pm\frac{1}{2}}) = K_2^{\pm\frac{1}{2}}$, $\Xi(K_2^{\pm\frac{1}{2}}) = K_3^{\pm\frac{1}{2}}$, $\Xi(K_3^{\pm\frac{1}{2}}) = K_0^{\pm\frac{1}{2}}$, $\Xi(K_0^{\pm\frac{1}{2}}) = K_1^{\pm\frac{1}{2}}$, $\Xi(E_1) = E_2$, $\Xi(E_2) = E_3$, $\Xi(E_3) = E_0$, $\Xi(E_0) = E_1$, $\Xi(F_1) = F_2$, $\Xi(F_2) = F_3$, $\Xi(F_3) = F_0$, $\Xi(F_0) = F_1$. (See [Y1].) Then we see the following.

Theorem 2.3. *Keep the notation as in the Theorem 2.2. Then the defining relations of* $U_q(\widehat{\mathrm{sl}}(2|2)) = U_q(\widehat{\mathrm{sl}}(2|2),$ 

) satisfied by the generators

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, K_3^{\pm\frac{1}{2}}, E_0, E_1, E_2, E_3, F_0, F_1, F_2, F_3,$$

are the following.

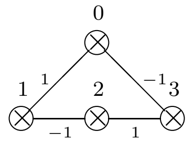
$$\begin{cases} \sigma^2 = 1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma = K_i^{\pm\frac{1}{2}},\ \sigma E_i \sigma = (-1)^{p(i)} E_i,\ \sigma F_i \sigma = (-1)^{p(i)} F_i, \\ K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1,\ K_i^{\frac{1}{2}} K_j^{\frac{1}{2}} = K_j^{\frac{1}{2}} K_i^{\frac{1}{2}} \\ K_i^{\frac{1}{2}} E_j K_i^{-\frac{1}{2}} = q^{\frac{a_{ij}}{2}} E_j,\ K_i^{\frac{1}{2}} F_j K_i^{-\frac{1}{2}} = q^{-\frac{a_{ij}}{2}} F_j, \\ E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \end{cases}$$

$$\begin{cases} E_0^2 = E_1^2 = E_2^2 = E_3^2 = 0, \\ [E_2, [[E_1, E_2]_{+,q}, E_3]_{-,q^{-1}}]_{+,1} = 0, \quad [E_3, [[E_2, E_3]_{+,q^{-1}}, E_0]_{-,q}]_{+,1} = 0, \\ [E_0, [[E_3, E_0]_{+,q}, E_1]_{-,q^{-1}}]_{+,1} = 0, \quad [E_1, [[E_0, E_1]_{+,q^{-1}}, E_2]_{-,q}]_{+,1} = 0, \end{cases}$$

$$\begin{cases} F_0^2 = F_1^2 = F_2^2 = F_3^2 = 0, \\ [F_2, [[F_1, F_2]_{+,q}, F_3]_{-,q^{-1}}]_{+,1} = 0, \quad [F_3, [[F_2, F_3]_{+,q^{-1}}, F_0]_{-,q}]_{+,1} = 0, \\ [F_0, [[F_3, F_0]_{+,q}, F_1]_{-,q^{-1}}]_{+,1} = 0, \quad [F_1, [[F_0, F_1]_{+,q^{-1}}, F_2]_{-,q}]_{+,1} = 0, \end{cases}$$

$$\left\{ \begin{array}{l} [E_1, \Big(ad_{-,1}([E_2, [E_1, [E_3, E_0]_{+,q}]_{-,q^{-1}}]_{+,1})\Big)^m (E_3)]_{+,1} = 0, \\ [F_1, \Big(ad_{-,1}([F_2, [F_1, [F_3, F_0]_{+,q}]_{-,q^{-1}}]_{+,1})\Big)^m (F_3)]_{+,1} = 0, \\ [E_2, \Big(ad_{-,1}([E_3, [E_2, [E_0, E_1]_{+,q^{-1}}]_{-,q}]_{+,1})\Big)^m (E_0)]_{+,1} = 0, \\ [F_2, \Big(ad_{-,1}([F_3, [F_2, [F_0, F_1]_{+,q^{-1}}]_{-,q}]_{+,1})\Big)^m (F_0)]_{+,1} = 0, \end{array} \right\} \textit{for } m \geq 1$$

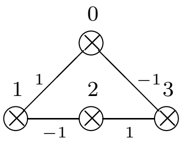
(see above for the notation $ad_{-,1}$*)* □

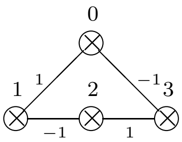
Theorem 2.4. *Keep the notation as in the Theorem 2.3. For* $1 \leq i \leq 3$ *and* $m \geq 1$, *define the elements* $\tilde{h}_{i,m}$, $\tilde{h}_{i,m} \in U_q(\widehat{\mathrm{sl}}(2|2)$, 

$)$ *by*

$$\tilde{h}_{i,m} = (-1)^{im}[E_i, T^m_{\omega_i}(K_i^{-1}F_i)]_{+,1} - (q-q^{-1}) \sum_{k=1}^{m-1} \frac{k(-1)^{i(m-k)}}{m} \tilde{h}_{i,k}[E_i, T^{m-k}_{\omega_i}(K_i^{-1}F_i)]_{+,1},$$

$$\tilde{h}_{i,-m} = (-1)^{im}[F_i, T^m_{\omega_i}(E_iK_i)]_{+,1} + (q-q^{-1}) \sum_{k=1}^{m-1} \frac{k(-1)^{i(m-k)}}{m} \tilde{h}_{i,-k}[F_i, T^{m-k}_{\omega_i}(E_iK_i)]_{+,1}.$$

Then $U_q(\mathbf{A}^{(1)}(1,1)$, 

$)$ *is the quotient algebra* $U_q(\widehat{\mathrm{sl}}(2|2)$, 

$)/J$, *where* J *is the ideal generated by*

$$\tilde{h}_{1,m} + \tilde{h}_{3,m}$$

and

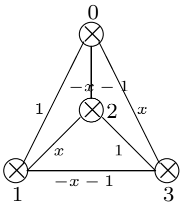
$$\tilde{h}_{1,-m} + \tilde{h}_{3,-m}$$

for $m \geq 1$.

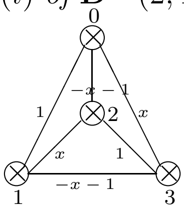
$\square$

Remark. In my paper, H. Yamane: Errata to "On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras", Publ. RIMS Kyoto Univ., 37 (2001), 615–619, there has still existed misprint. The (QS4)(19) should be modified into

(QS4)(19) $[E_1, T^{-m}_{\omega_3}(E_3)] = 0$, $Z_1([E_1, T^{-m}_{\omega_3}(E_3)]) = 0$ $(m \geq 1)$ *if* $(\mathcal{E}, \Pi, p)$ *is an unusual datum of* $(\mathbf{A}(1,1)^{(1)})^{\mathcal{H}}$ *type.*

2.3. Defining relations of $U_q(\mathbf{D}^{(1)}(2,1;x)$, 

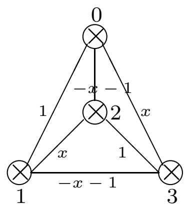
) $(x \neq 0, -1)$

Theorem 2.5. *Assume $x \neq 0, -1$. Define the Cartan matrix $A = (a_{ij})$ and the parity $p(i)$ of $\mathbf{D}^{(1)}(2,1;x)$, the affine Lie superalgebra whose Dynkin diagram is* 

, by the following.

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} := \begin{pmatrix} 0 & 1 & -x-1 & x \\ 1 & 0 & x & -x-1 \\ -x-1 & x & 0 & 1 \\ x & -x-1 & 1 & 0 \end{pmatrix}$$

$$p(0) := p(1) := p(2) := p(3) := 1.$$

Then the defining relations of $U_q(\mathbf{D}^{(1)}(2,1;x)) = U_q(\mathbf{D}^{(1)}(2,1;x)$, 

) *satisfied by the generators*

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, K_3^{\pm\frac{1}{2}}, E_0, E_1, E_2, E_3, F_0, F_1, F_2, F_3,$$

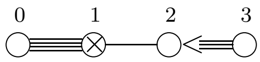
are the following.

$$\begin{cases} \sigma^2 = 1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma = K_i^{\pm\frac{1}{2}},\ \sigma E_i \sigma = (-1)^{p(i)} E_i,\ \sigma F_i \sigma = (-1)^{p(i)} F_i, \\ K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1,\ K_i^{\frac{1}{2}} K_j^{\frac{1}{2}} = K_j^{\frac{1}{2}} K_i^{\frac{1}{2}} \\ K_i^{\frac{1}{2}} E_j K_i^{-\frac{1}{2}} = q^{\frac{a_{ij}}{2}} E_j,\ K_i^{\frac{1}{2}} F_j K_i^{-\frac{1}{2}} = q^{-\frac{a_{ij}}{2}} F_j, \\ E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \end{cases}$$

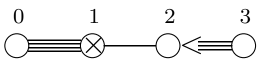
$$\begin{cases} E_0^2 = E_1^2 = E_2^2 = E_3^2 = 0, \\ \frac{q^{x+1}-q^{-x-1}}{q-q^{-1}}[[E_1, E_2]_{+,q^{-x}}, E_3]_{-,q^x} + \frac{q^x-q^{-x}}{q-q^{-1}}[[E_1, E_3]_{+,q^{x+1}}, E_2]_{-,q^x} = 0, \\ \frac{q^{x+1}-q^{-x-1}}{q-q^{-1}}[[E_0, E_1]_{+,q^{-1}}, E_2]_{-,q} + [[E_0, E_2]_{+,q^{x+1}}, E_1]_{-,q^{-x-1}} = 0, \\ \frac{q^x-q^{-x}}{q-q^{-1}}[[E_0, E_1]_{+,q^{-1}}, E_3]_{-,q} - [[E_0, E_3]_{+,q^{-x}}, E_1]_{-,q^x} = 0, \\ \frac{q^x-q^{-x}}{q-q^{-1}}[[E_0, E_2]_{+,q^{x+1}}, E_3]_{-,q^{-x-1}} + \frac{q^{x+1}-q^{-x-1}}{q-q^{-1}}[[E_0, E_3]_{+,q^{-x}}, E_2]_{-,q^x} = 0, \end{cases}$$

$$\begin{cases} F_0^2 = F_1^2 = F_2^2 = F_3^2 = 0, \\ \frac{q^{x+1}-q^{-x-1}}{q-q^{-1}}[[F_1, F_2]_{+,q^{-x}}, F_3]_{-,q^x} + \frac{q^x-q^{-x}}{q-q^{-1}}[[F_1, F_3]_{+,q^{x+1}}, F_2]_{-,q^x} = 0, \\ \frac{q^{x+1}-q^{-x-1}}{q-q^{-1}}[[F_0, F_1]_{+,q^{-1}}, F_2]_{-,q} + [[F_0, F_2]_{+,q^{x+1}}, F_1]_{-,q^{-x-1}} = 0, \\ \frac{q^x-q^{-x}}{q-q^{-1}}[[F_0, F_1]_{+,q^{-1}}, F_3]_{-,q} - [[F_0, F_3]_{+,q^{-x}}, F_1]_{-,q^x} = 0, \\ \frac{q^x-q^{-x}}{q-q^{-1}}[[F_0, F_2]_{+,q^{x+1}}, F_3]_{-,q^{-x-1}} + \frac{q^{x+1}-q^{-x-1}}{q-q^{-1}}[[F_0, F_3]_{+,q^{-x}}, F_2]_{-,q^x} = 0, \end{cases}$$

□

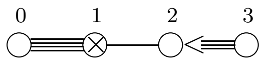
2.4. Defining relations of $U_q(\mathbf{G}^{(1)}(3),$ )

Theorem 2.6. *Define the Cartan matrix $A = (a_{ij})$ and the parity $p(i)$ of $\mathbf{G}^{(1)}(3)$, the affine Lie superalgebra whose Dynkin diagram is*

 *, by the following.*

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} := \begin{pmatrix} -8 & 4 & 0 & 0 \\ 4 & 0 & -1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{pmatrix}$$

$$p(0) := p(2) := p(3) := 0, p(1) := 1.$$

Then the defining relations of $U_q(\mathbf{G}^{(1)}(3)) = U_q(\mathbf{G}^{(1)}(3),$ ) *satisfied by the generators*

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, K_3^{\pm\frac{1}{2}}, E_0, E_1, E_2, E_3, F_0, F_1, F_2, F_3,$$

are the following.

$$\begin{cases} \sigma^2=1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma=K_i^{\pm\frac{1}{2}},\ \sigma E_i\sigma=(-1)^{p(i)}E_i,\ \sigma F_i\sigma=(-1)^{p(i)}F_i,\\ K_i^{\frac{1}{2}}K_i^{-\frac{1}{2}}=K_i^{-\frac{1}{2}}K_i^{\frac{1}{2}}=1,\ K_i^{\frac{1}{2}}K_j^{\frac{1}{2}}=K_j^{\frac{1}{2}}K_i^{\frac{1}{2}}\\ K_i^{\frac{1}{2}}E_jK_i^{-\frac{1}{2}}=q^{\frac{a_{ij}}{2}}E_j,\ K_i^{\frac{1}{2}}F_jK_i^{-\frac{1}{2}}=q^{-\frac{a_{ij}}{2}}F_j,\\ E_iF_j-(-1)^{p(i)p(j)}F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}} \end{cases}$$

$$\begin{cases} E_1^2=0,[E_0,E_2]_{-,1}=0,[E_0,E_3]_{-,1}=0,[E_1,E_3]_{-,1}=0,\\ [E_0,[E_0,E_1]_{-,q^{-4}}]_{-,q^4}=0,[E_2,[E_2,E_1]_{-,q}]_{-,q^{-1}}=0,\\ [E_2,[E_2,[E_2,[E_2,E_3]_{-,q^3}]_{-,q}]_{-,q^{-1}}]_{-,q^{-3}}=0,[E_3,[E_3,E_2]_{-,q^3}]_{-,q^{-3}}=0,\\ \frac{q^2-q^{-2}}{q-q^{-1}}[[[[[E_0,E_1]_{+,q^{-4}},E_2]_{-,q},E_3]_{-,q^3},E_1]_{+,q^{-3}},E_2]_{-,q^3}\\ \quad -\frac{q^3-q^{-3}}{q-q^{-1}}[[[[[E_0,E_1]_{+,q^{-4}},E_2]_{-,q},E_3]_{-,q^3},E_2]_{+,q^2},E_1]_{-,q^{-2}}=0, \end{cases}$$

$$\begin{cases} F_1^2=0,[F_0,F_2]_{-,1}=0,[F_0,F_3]_{-,1}=0,[F_1,F_3]_{-,1}=0,\\ [F_0,[F_0,F_1]_{-,q^{-4}}]_{-,q^4}=0,[F_2,[F_2,F_1]_{-,q}]_{-,q^{-1}}=0,\\ [F_2,[F_2,[F_2,[F_2,F_3]_{-,q^3}]_{-,q}]_{-,q^{-1}}]_{-,q^{-3}}=0,[F_3,[F_3,F_2]_{-,q^3}]_{-,q^{-3}}=0,\\ \frac{q^2-q^{-2}}{q-q^{-1}}[[[[[F_0,F_1]_{+,q^{-4}},F_2]_{-,q},F_3]_{-,q^3},F_1]_{+,q^{-3}},F_2]_{-,q^3}\\ \quad -\frac{q^3-q^{-3}}{q-q^{-1}}[[[[[F_0,F_1]_{+,q^{-4}},F_2]_{-,q},F_3]_{-,q^3},F_2]_{+,q^2},F_1]_{-,q^{-2}}=0. \end{cases}$$

□

2.5 The defining relations of $U_q(\mathbf{B}^{(1)}(1,1), \overset{0}{\bigcirc}\Longrightarrow\overset{1}{\otimes}\Longrightarrow\overset{2}{\bigcirc})$

Theorem 2.7. *Define the Cartan matrix $A=(a_{ij})$ and the parity $p(i)$ of $\mathbf{B}^{(1)}(1,1)$, the affine Lie superalgebra whose Dynkin diagram is $\overset{0}{\bigcirc}\Longrightarrow\overset{1}{\otimes}\Longrightarrow\overset{2}{\bigcirc}$, by the following.*

$$A=\begin{pmatrix} a_{00} & a_{01} & a_{02}\\ a_{10} & a_{11} & a_{12}\\ a_{20} & a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} -4 & 2 & 0\\ 2 & 0 & -1\\ 0 & -1 & 1 \end{pmatrix}$$

$$p(0)=p(2)=1,\ p(1)=0.$$

Then the defining relations of $U_q(\mathbf{B}^{(1)}(1,1)) = U_q(\mathbf{B}^{(1)}(1,1), \overset{0}{\bigcirc}\Longrightarrow\overset{1}{\otimes}\Longrightarrow\overset{2}{\bigcirc})$ *satisfied by the generators*

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, E_0, E_1, E_2, F_0, F_1, F_2,$$

are the following.

$$\begin{cases} \sigma^2 = 1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma = K_i^{\pm\frac{1}{2}},\ \sigma E_i \sigma = (-1)^{p(i)} E_i,\ \sigma F_i \sigma = (-1)^{p(i)} F_i, \\ K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1,\ K_i^{\frac{1}{2}} K_j^{\frac{1}{2}} = K_j^{\frac{1}{2}} K_i^{\frac{1}{2}} \\ K_i^{\frac{1}{2}} E_j K_i^{-\frac{1}{2}} = q^{\frac{a_{ij}}{2}} E_j,\ K_i^{\frac{1}{2}} F_j K_i^{-\frac{1}{2}} = q^{-\frac{a_{ij}}{2}} F_j, \\ E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \end{cases}$$

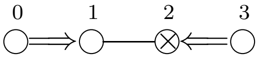
$$\begin{cases} E_1^2 = 0,\ [E_0, E_2]_{-,1} = 0,\ [E_0, [E_0, E_1]_{-,q^2}]_{-,q^{-2}} = 0, \\ [E_2, [E_2, [E_2, E_1]_{-,q}]_{-,1}]_{-,q^{-1}} = 0, \\ \Big([[E_2, E_1]_{-,q}, [[E_2, E_1]_{-,q}, [[E_2, E_1]_{-,q}, E_0]_{-,q^{-2}}]_{+,q^{-1}}]_{-,1} \\ \quad +(q - 1 + q^{-1})[E_1, [[E_2, E_1]_{-,q}, [E_2, [E_2, [E_1, E_0]_{-,q}]_{-,q^{-2}}]_{-,q^2}]_{-,q^{-1}}]_{-,q^3}\Big) = 0, \end{cases}$$

$$\begin{cases} F_1^2 = 0,\ [F_0, F_2]_{-,1} = 0,\ [F_0, [F_0, F_1]_{-,q^2}]_{-,q^{-2}} = 0, \\ [F_2, [F_2, [F_2, F_1]_{-,q}]_{-,1}]_{-,q^{-1}} = 0, \\ \Big([[F_2, F_1]_{-,q}, [[F_2, F_1]_{-,q}, [[F_2, F_1]_{-,q}, F_0]_{-,q^{-2}}]_{+,q^{-1}}]_{-,1} \\ \quad +(q - 1 + q^{-1})[F_1, [[F_2, F_1]_{-,q}, [F_2, [F_2, [F_1, F_0]_{-,q}]_{-,q^{-2}}]_{-,q^2}]_{-,q^{-1}}]_{-,q^3}\Big) = 0, \end{cases}$$

$\square$

2.6 The defining relations of $U_q(\mathbf{A}^{(2)}(1,3), \overset{0}{\bigcirc}\Longrightarrow\overset{1}{\bigcirc}\!\!-\!\!-\!\!-\overset{2}{\otimes}\Longleftarrow\overset{3}{\bigcirc}\ )$

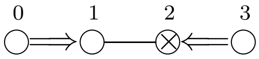
Theorem 2.8. *Define the Cartan matrix* $A = (a_{ij})$ *and the parity* $p(i)$ *of* $\mathbf{A}^{(2)}(1,3)$*, the affine Lie superalgebra whose Dynkin diagram is*



, by the following.

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} := \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$$p(0) := p(1) := p(3) := 0, p(2) := 1.$$

Then the defining relations of $U_q(\mathbf{A}^{(2)}(1,3)) = U_q(\mathbf{A}^{(2)}(1,3),$ 

$)$ *satisfied by the generators*

$$\sigma, K_0^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}, K_2^{\pm\frac{1}{2}}, K_3^{\pm\frac{1}{2}}, E_0, E_1, E_2, E_3, F_0, F_1, F_2, F_3,$$

are the following.

$$\begin{cases} \sigma^2 = 1,\ \sigma K_i^{\pm\frac{1}{2}}\sigma = K_i^{\pm\frac{1}{2}},\ \sigma E_i \sigma = (-1)^{p(i)} E_i,\ \sigma F_i \sigma = (-1)^{p(i)} F_i, \\ K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1,\ K_i^{\frac{1}{2}} K_j^{\frac{1}{2}} = K_j^{\frac{1}{2}} K_i^{\frac{1}{2}} \\ K_i^{\frac{1}{2}} E_j K_i^{-\frac{1}{2}} = q^{\frac{a_{ij}}{2}} E_j,\ K_i^{\frac{1}{2}} F_j K_i^{-\frac{1}{2}} = q^{-\frac{a_{ij}}{2}} F_j, \\ E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \end{cases}$$

$$\begin{cases} E_2^2 = 0, [E_0, E_2]_{-,1} = 0, [E_0, E_3]_{-,1} = 0, [E_1, E_3]_{-,1} = 0, \\ [E_0, [E_0, E_1]_{-,q^2}]_{-,q^{-2}} = 0, [E_1, [E_1, [E_1, E_0]_{-,q^2}]_{-,1}]_{-,q^{-2}} = 0, \\ [E_1, [E_1, E_2]_{-,q}]_{-,q^{-1}} = 0, [E_3, [E_3, E_2]_{-,q^{-2}}]_{-,q^2} = 0, \\ \Big([[[E_2, [E_3, [E_2, E_1]_{-,q}]_{-,q^{-2}}]_{+,q^{-1}}, [E_2, [E_3, [E_2, [E_1, E_0]_{-,q^2}]_{-,q}]_{-,q^{-2}}]_{+,q^{-1}}]_{+,1}]_{-,q^2} \\ \quad -(q + q^{-1})\Big([[[E_2, E_1]_{-,q}, [[E_3, [E_2, E_1]_{-,q}]_{+,q^{-2}}, \\ \qquad [E_2, [E_3, [E_2, [E_1, E_0]_{-,q^2}]_{-,q}]_{-,q^{-2}}]_{+,q^{-1}}]_{-,q}]_{-,q^{-1}}\Big)\Big) = 0, \end{cases}$$

$$\begin{cases} F_2^2=0, [F_0,F_2]_{-,1}=0, [F_0,F_3]_{-,1}=0, [F_1,F_3]_{-,1}=0, \\ [F_0,[F_0,F_1]_{-,q^2}]_{-,q^{-2}}=0, [F_1,[F_1,[F_1,F_0]_{-,q^2}]_{-,1}]_{-,q^{-2}}=0, \\ [F_1,[F_1,F_2]_{-,q}]_{-,q^{-1}}=0, [F_3,[F_3,F_2]_{-,q^{-2}}]_{-,q^2}=0, \\ \Big([[[F_2,[F_3,[F_2,F_1]_{-,q}]_{-,q^{-2}}]_{+,q^{-1}}, [F_2,[F_3,[F_2,[F_1,F_0]_{-,q^2}]_{-,q}]_{-,q^{-2}}]_{+,q^{-1}}]_{+,1}]_{-,q^2} \\ \quad -(q+q^{-1})\Big([[[F_2,F_1]_{-,q}, [[F_3,[F_2,F_1]_{-,q}]_{+,q^{-2}}, \\ \qquad [F_2,[F_3,[F_2,[F_1,F_0]_{-,q^2}]_{-,q}]_{-,q^{-2}}]_{+,q^{-1}}]_{-,q}]_{-,q^{-1}}\Big)\Big)=0, \end{cases}$$

□

Remark. There exist misprints in (QS4)(9). In (QS4)(9), [should be replaced by ⟦, and] should be replaced by ⟧.

2.7 The defining relations of $U_q(\widehat{\mathrm{sl}}(m+1|n+1))$ with $m+n+2 \geq 5$ or with $m=2$, $n=0$

Unless $m=n=1$, $\widehat{\mathrm{sl}}(m+1|n+1)$ is the universal central extension of $\mathbf{A}^{(1)}(m,n)$. Notice that $\widehat{\mathrm{sl}}(m+1|n+1) \neq \mathbf{A}^{(1)}(m,n)$ if and only if $m=n$.

Notation: Let $f:\mathbb{Z}\to\mathbb{Z}/(m+n+2)\mathbb{Z}$ be the natural surjective additive group homomorphism, where $\mathbb{Z}/(m+n+2)\mathbb{Z}$ is the cyclic group of order $m+n+2$. In particular, $f(i+m+n+2)=f(i)$. Let V be an $m+n+3$-dimensional $\mathbb{C}$-vector space with basis $\{\varepsilon_{f(i)}|i\in\mathbb{Z}\}\cup\{\delta\}$. Let $(\ ,\ )$ be a symmetric bilinear form on V such that $(\varepsilon_{f(i)},\varepsilon_{f(i)})\in\{1,-1\}$, and $(\varepsilon_{f(i)},\varepsilon_{f(j)})=0$ for $f(i)\neq f(j)$, and $(\delta,\delta)=(\delta,\varepsilon_{f(i)})=0$. Assume that the number of the elements $\varepsilon_{f(i)}$ such that $(\varepsilon_{f(i)},\varepsilon_{f(i)})=1$ is exactly $m+1$. Define $m+n+2$ elements $\{\alpha_{f(i)}|i\in\mathbb{Z}\}$ of V by $\alpha_{f(i)}=\varepsilon_{f(i)}-\varepsilon_{f(i+1)}+\delta_{f(i),f(0)}\delta$ (Here $\delta_{f(i)f(0)}$ is the Kronecker delta, i.e., $\delta_{f(0),f(0)}=1$ and $\delta_{f(0),f(i)}=0$ $(f(0)\neq f(i))$). Let $a_{ij}:=(\alpha_{f(i)},\alpha_{f(j)})$ and $p(i):=\frac{(\alpha_{f(i)},\alpha_{f(i)})^2}{4}\in\{0,1\}$.

Theorem 2.9. *Keep the notation as above. Assume that $m+n+2\geq 5$ or assume that $m+n+2=4$ and $m\neq n$. Then the defining relations* *of* $U_q(\widehat{\mathrm{sl}}(m+1|n+1),f)=U_q(\widehat{\mathrm{sl}}(m+1|n+1))$ *satisfied by the generators*

$$\sigma, K_{f(i)}^{\pm\frac{1}{2}}, E_{f(i)}, F_{f(i)} \quad (i\in\mathbb{Z})$$

are the following.

$$\begin{cases} \sigma^2=1,\ \sigma K_{f(i)}^{\pm\frac{1}{2}}\sigma=K_{f(i)}^{\pm\frac{1}{2}},\ \sigma E_{f(i)}\sigma=(-1)^{p(i)}E_{f(i)},\ \sigma F_{f(i)}\sigma=(-1)^{p(i)}F_{f(i)},\\ K_{f(i)}^{\frac{1}{2}}K_{f(i)}^{-\frac{1}{2}}=1,\ K_{f(i)}^{\frac{1}{2}}K_{f(j)}^{\frac{1}{2}}=K_{f(j)}^{\frac{1}{2}}K_{f(i)}^{\frac{1}{2}}\\ K_{f(i)}^{\frac{1}{2}}E_{f(j)}K_{f(i)}^{-\frac{1}{2}}=q^{\frac{a_{ij}}{2}}E_{f(j)},\ K_{f(i)}^{\frac{1}{2}}F_{f(j)}K_{f(i)}^{-\frac{1}{2}}=q^{-\frac{a_{ij}}{2}}F_{f(j)},\\ E_{f(i)}F_{f(j)}-(-1)^{p(i)p(j)}F_{f(j)}E_{f(i)}=\delta_{ij}\frac{K_{f(i)}-K_{f(i)}^{-1}}{q-q^{-1}} \end{cases}$$

$$\begin{cases} E_{f(i)}E_{f(j)}-(-1)^{p(i)p(j)}E_{f(i)}E_{f(j)}=0 \qquad \text{if } a_{ij}=0 \text{ and } f(i)\neq f(j),\\ E_{f(i)}^2=0 \qquad \text{if } a_{ii}=0,\\ [E_{f(i)},[E_{f(i)},E_{f(i+1)}]_{-,q}]_{-,q^{-1}}=0, [E_{f(i)},[E_{f(i)},E_{f(i-1)}]_{-,q}]_{-,q^{-1}} \qquad \text{if } a_{ii}\neq 0,\\ \Big(E_{f(i)}\Big(E_{f(i-1)}E_{f(i)}E_{f(i+1)}-(-1)^{p(i-1)}q^{-a_{i-1,i}}E_{f(i)}E_{f(i-1)}E_{f(i+1)}\\ \qquad -(-1)^{p(i+1)(1+p(i-1))}q^{-a_{i+1,i}}E_{f(i+1)}E_{f(i-1)}E_{f(i)}\\ \qquad +(-1)^{p(i-1)+p(i+1)+p(i-1)p(i+1)}E_{f(i+1)}E_{f(i)}E_{f(i-1)}\Big)\\ +(-1)^{p(i-1)+p(i+1)}\Big(E_{f(i-1)}E_{f(i)}E_{f(i+1)}-(-1)^{p(i-1)}q^{-a_{i-1,i}}E_{f(i)}E_{f(i-1)}E_{f(i+1)}\\ \qquad -(-1)^{p(i+1)(1+p(i-1))}q^{-a_{i+1,i}}E_{f(i+1)}E_{f(i-1)}E_{f(i)}\\ \qquad +(-1)^{p(i-1)+p(i+1)+p(i-1)p(i+1)}E_{f(i+1)}E_{f(i)}E_{f(i-1)}\Big)E_{f(i)}\Big)=0 \qquad \text{if } a_{ii}=0 \end{cases}$$

$$\begin{cases} F_{f(i)}F_{f(j)} - (-1)^{p(i)p(j)}F_{f(i)}F_{f(j)} = 0 \qquad \textit{if } a_{ij} = 0 \textit{ and } f(i) \neq f(j), \\ F_{f(i)}^2 = 0 \qquad \textit{if } a_{ii} = 0, \\ [F_{f(i)}, [F_{f(i)}, F_{f(i+1)}]_{-,q}]_{-,q^{-1}} = 0, [F_{f(i)}, [F_{f(i)}, F_{f(i-1)}]_{-,q}]_{-,q^{-1}} \qquad \textit{if } a_{ii} \neq 0, \\ \Big(F_{f(i)}\Big(F_{f(i-1)}F_{f(i)}F_{f(i+1)} - (-1)^{p(i-1)}q^{-a_{i-1,i}}F_{f(i)}F_{f(i-1)}F_{f(i+1)} \\ \quad -(-1)^{p(i+1)(1+p(i-1))}q^{-a_{i+1,i}}F_{f(i+1)}F_{f(i-1)}F_{f(i)} \\ \quad +(-1)^{p(i-1)+p(i+1)+p(i-1)p(i+1)}F_{f(i+1)}F_{f(i)}F_{f(i-1)}\Big) \\ +(-1)^{p(i-1)+p(i+1)}\Big(F_{f(i-1)}F_{f(i)}F_{f(i+1)} - (-1)^{p(i-1)}q^{-a_{i-1,i}}F_{f(i)}F_{f(i-1)}F_{f(i+1)} \\ \quad -(-1)^{p(i+1)(1+p(i-1))}q^{-a_{i+1,i}}F_{f(i+1)}F_{f(i-1)}F_{f(i)} \\ \quad +(-1)^{p(i-1)+p(i+1)+p(i-1)p(i+1)}F_{f(i+1)}F_{f(i)}F_{f(i-1)}\Big) F_{f(i)}\Big) = 0 \qquad \textit{if } a_{ii} = 0 \end{cases}$$

□

2.8 A PBW theorem (or a topological freeness) of $U_q(\widehat{\mathrm{sl}}(m+1|n+1))$ with $m+n+2 \geq 5$ or with $m=2$, $n=0$

Lemma 2.10 (*Vector representation*). *Assume $m+n \geq 1$. Denote $U_q = U_q(\widehat{\mathrm{sl}}(m+1|n+1), f)$. Let $\mathbb{F} := \mathbb{C}(q^{\frac{1}{2}})$. Let V' be the subspace of V spanned by $\varepsilon_{f(i)}$'s. Let $M := \mathrm{End}_{\mathbb{K}}(V' \otimes_{\mathbb{C}} \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[t, t^{-1}]$. Then there exists a homomorphism $\rho : U_q \to M$ such that $\rho(\sigma)\varepsilon_{f(i)} = (\varepsilon_{f(i)}, \varepsilon_{f(i)})\varepsilon_{f(i)}$, $\rho(K_{f(i)}^{\pm\frac{1}{2}})\varepsilon_{f(j)} = q^{\pm\frac{1}{2}(\alpha_{f(i)}, \varepsilon_{f(j)})}\varepsilon_{f(j)}$, $\rho(E_{f(i)})\varepsilon_{f(j)} = \delta_{f(i+1),f(j)}t^{\delta_{f(i),f(0)}}(\varepsilon_{f(i)}, \varepsilon_{f(i)})\varepsilon_{f(i)}$ and $\rho(F_{f(i)})\varepsilon_{f(j)} = \delta_{f(i),f(j)}t^{-\delta_{f(i),f(0)}}\varepsilon_{f(i+1)}$.*

Let

$$(R^{\mathrm{re}})^+ := \{\varepsilon_{f(i)} - \varepsilon_{f(j)} + k\delta \in V \,|\, k \in \mathbb{Z}_+,\ 1 \leq j - i \leq m+n+1\,\},$$

$(R^{\mathrm{re}}_{\mathrm{odd}})^+ := \{\beta \in (R^{\mathrm{re}})^+ \,|\, (\beta, \beta) = 0\,\}$, $(R^{\mathrm{re}}_{\mathrm{even}})^+ := (R^{\mathrm{re}})^+ \setminus (R^{\mathrm{re}}_{\mathrm{odd}})^+$ and

$$(\widetilde{R}^{\mathrm{im}})^+ := \{(k\delta, i) \in V \times \{1, \ldots, m+n+1\} \,|\, k \in \mathbb{N}\,\}.$$

Theorem 2.11 (*A PBW theorem*). *Keep the notation of Lemma 2.10. Let* U_q^+ *(respectively* U_q^-*, respectively* U_q^0*) be the subalgebra of* U_q *generated by* $E_{f(i)}$*'s (respectively* $F_{f(i)}$*'s, respectively* σ *and* $K_{f(i)}^{\pm\frac{1}{2}}$*'s). Then we have the following.*

(1) $U_q \cong U_q^+ \otimes U_q^0 \otimes U_q^-$ *as* $\mathbb{K}$*-vector spaces, and* $\{K_{f(0)}^{\frac{x_0}{2}} \cdots K_{f(m+n+1)}^{\frac{x_{m+n+1}}{2}} \sigma^c \,|\, x_i \in \mathbb{Z},\, c \in \{0,1\}\}$ *form a* $\mathbb{K}$*-basis of* U_q^0*. Moreover the* $\mathbb{K}$*-algebra* U_q^+ *(respectively* U_q^-*) is also presented by the generators* $E_{f(i)}$ *(respectively* $F_{f(i)}$*) and the same defining relations in Theorems 2.1, 2.3 and 2.9.*

(2) For each $\beta = \varepsilon_{f(i)} - \varepsilon_{f(j)} + k\delta \in (R^{\mathrm{re}})^+$*, choose an* $E_\beta \in U_q^+$ *so that* $\rho(E_\beta) = E_{f(i)f(j)} \otimes t^k$*. For each* $(k\delta, i) \in (\widetilde{R}^{\mathrm{im}})^+$*, choose an* $E_{(k\delta,i)} \in U_q^+$ *so that* $\rho(E_{(k\delta,i)}) = ((\varepsilon_{f(i)}, \varepsilon_{f(i)})E_{f(i)f(i)} - (\varepsilon_{f(i+1)}, \varepsilon_{f(i+1)})E_{f(i+1)f(i+1)}) \otimes t^k$*. Then the elements*

$$\prod_{\mu \in (R^{\mathrm{re}})^+ \cup (\widetilde{R}^{\mathrm{im}})^+} E_\mu^{x_\mu},$$

where $x_\beta \in \mathbb{Z}_+$ *if* $\beta \in (R^{\mathrm{re}}_{\mathrm{even}})^+$*;* $x_\beta \in \{0,1\}$ *if* $\beta \in (R^{\mathrm{re}}_{\mathrm{odd}})^+$*;* $x_{(k\delta,i)} \in \mathbb{Z}_+$ *if* $(k\delta, i) \in (\widetilde{R}^{\mathrm{im}})^+$*, form a* $\mathbb{K}$*-basis of* U_q^+*. Here the product is in a predetermined total order.*

Proof. (1) This can be proved in a standard way.

(2) Notice that the E_μ exists for every $\mu \in (R^{\mathrm{re}})^+ \cup (\widetilde{R}^{\mathrm{im}})^+$. We can show in the same manner as in [Y1] that a standard coproduct $\Delta : U_q \to U_q \otimes U_q$ exists. Then, by using induction and by using the homomorphisms $\rho^n \circ \Delta^{(n-1)} : U_q \to M^{\otimes n}$ for all n, the elements in the statement are linearly independent.

Let U_1^+ be the $\mathbb{C}$-algebra defined with the generators $E_{f(i)}$ and the defining relations obtained from those of U_q in Theorems 2.1, 2.3 and 2.9 by putting $q = 1$. We have known the fact that U_1^+ is isomorphic to the universal enveloping algebras of a positive part of $\widehat{\mathrm{sl}}(m+1|n+1)$, which implies that

$$\dim_{\mathbb{K}}(U_q^+)_\mu \leq \dim_{\mathbb{C}}(U_1^+)_\mu \tag{2.1}$$

for every $\mu \in \oplus_{i=0}^{m+n+1}\mathbb{Z}\alpha_{f(i)}$, where $(U_q^+)_\mu$ and $(U_1^+)_\mu$ denote the weight spaces of the weight μ. However, by the fact shown in the last paragraph, we see that the inequality in (2.1) is indeed the equality. This completes the proof. $\square$.

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