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Gravitational Waves

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Contents

Gravitational Waves in General Relativity

Einstein Equation

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$$

μ, ν : 0 (time),1,2,3 (space)

- 10 partial differentiation simultaneous equation.
- 10 parts of metric tensor are coupled with each other. (16 is decreased to 10 because of symmetry.)
- 4 for conservation equations, 6 for motion equations
- Non-linear.

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$A_{0\,0\sim3}$: 保存則に対応

A_{nm} : 運動方程式に対応

Analogy with Electromagnetism

- **Maxwell Equation**

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)A^\nu=-\mu_0 i^\nu,\qquad \frac{\partial A^\nu}{\partial x^\nu}=0 : Lorenz\ Condition$$

- **Retarded Potential**

$$A^\nu=\frac{\mu_0}{4\pi}\int\frac{i^\nu\left(t-\frac{|r-r'|}{c},r'\right)}{|r-r'|}dV'$$

- **Electrical Dipole radiation**

$$W=\frac{\mu_0\omega^4}{12\pi c}|P_\omega|^2$$

ω : angular frequency of EMW

Gravitational Waves

(1) Small ripples (perturbations, $h_{\mu\nu}$) of a flat space-time ($\eta_{\mu\nu}$)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (|h_{\mu\nu}| \ll 1)$$

(2) Linearized Einstein Equation

1 : Strictly speaking, $h_{\mu\nu}$ is not a Tensor. However, we can use $\eta_{\mu\nu}$ to arrange $h_{\mu\nu}$ and its partial differentiation.

$$h_{\mu\nu}{}^{,\alpha} = \eta^{\alpha\beta} h_{\mu\nu,\beta}$$

$$\left(\frac{\partial A}{\partial x^\alpha} = A_{,\alpha} \,, \frac{\partial A}{\partial x_\alpha} = A^{,\alpha} \,, \frac{\partial^2 A}{\partial x^\alpha \partial x^\beta} = A_{,\alpha\beta} \,, \frac{\partial^2 A}{\partial x^\alpha \partial x_\beta} = A^{,\beta}{}_{,\alpha} \right)$$

Einstein Equation Linearization

(2-2) Christoffel symbols

$$\Gamma^{\alpha}{}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right) \approx \frac{1}{2}\eta^{\alpha\mu}\left(h_{\mu\beta,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu}\right)$$

$$\equiv \frac{1}{2}\left(h^{\alpha}{}_{\beta,\gamma} + h^{\alpha}{}_{\gamma,\beta} - h_{\beta\gamma}{}^{,\alpha}\right)$$

(2-3) Ricci Tensor

$$R^{\mu}{}_{\alpha\beta\gamma} \equiv \Gamma^{\mu}{}_{\alpha\gamma,\beta} - \Gamma^{\mu}{}_{\alpha\beta,\gamma} + O(\Gamma^2)$$

$$R_{\mu\nu}\left(= R^{\alpha}{}_{\mu\alpha\nu}\right) \equiv \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + O(h^2)$$

$$\approx \frac{1}{2}\left\{h^{\alpha}{}_{\nu,\mu\alpha} + h_{\mu\alpha}{}^{,\alpha}{}_{\nu} - h_{\mu\nu}{}^{,\alpha}{}_{\alpha} + h^{\alpha}{}_{\alpha,\mu\nu}\right\}$$

(2-3) Ricci Scalar

$$R = R^{\beta}{}_{\beta}$$

$$\approx \frac{1}{2}\left\{h^{\alpha}{}_{\beta,\alpha}{}^{\beta} + h^{\beta}{}_{\alpha,\beta}{}^{\alpha} - h^{\beta}{}_{\beta,\alpha}{}^{\alpha} + h_{,\beta}{}^{\beta}\right\} = h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}$$

Einstein Equation Linearization

$$G_{\mu\nu} = \frac{1}{2}\left(-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{\nu} + \bar{h}_{\nu\alpha}{}^{,\alpha}{}_{\mu}\right)$$

$$\left(\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h,\quad \bar{h} = \bar{h}^{\alpha}{}_{\alpha} = -h(= -h^{\alpha}{}_{\alpha}),\quad h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right)$$

(2-4) Einstein Equation with Gauge Condition

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$$

Between two coordinates (x^{μ} to $x^{\mu'}$ $(h \rightarrow h')$ assuming infinitesimal transformation),

$$x^{\mu'} = x^{\mu} + \xi^{\mu}, (\xi^{\mu} \ll 1)$$

$$\eta_{\mu\nu} + h_{\mu'\nu'} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}}\frac{\partial x^{\beta}}{\partial x^{\nu'}}(\eta_{\alpha\beta} + h_{\alpha\beta}) \sim \frac{\partial x^{\alpha}}{\partial x^{\mu}}\left(1 - \frac{\partial \xi^{\mu}}{\partial x^{\mu}}\right)\frac{\partial x^{\beta}}{\partial x^{\nu}}\left(1 - \frac{\partial \xi^{\nu}}{\partial x^{\nu}}\right)(\eta_{\alpha\beta} + h_{\alpha\beta})$$

$(h'_{\mu\nu})$

$$= \left(\delta^{\alpha}_{\mu} - \delta^{\alpha}_{\mu}\frac{\partial \xi^{\mu}}{\partial x^{\mu}}\right)\left(\delta^{\beta}_{\nu} - \delta^{\beta}_{\mu}\frac{\partial \xi^{\nu}}{\partial x^{\nu}}\right)(\eta_{\alpha\beta} + h_{\alpha\beta})$$

$$= \left(\delta^{\alpha}_{\mu} - \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\right)\left(\delta^{\beta}_{\nu} - \frac{\partial \xi^{\beta}}{\partial x^{\nu}}\right)(\eta_{\alpha\beta} + h_{\alpha\beta})$$

Einstein Equation Linearization

$$\eta_{\mu\nu} + h_{\mu'\nu'} = \left(\delta^{\alpha}_{\mu} - \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\right)\left(\delta^{\beta}_{\nu} - \frac{\partial \xi^{\beta}}{\partial x^{\nu}}\right)(\eta_{\alpha\beta} + h_{\alpha\beta})$$

$(h'_{\mu\nu})$

$$\approx \eta_{\alpha\beta}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + h_{\alpha\beta}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}(\eta_{\alpha\beta}) - \frac{\partial \xi^{\beta}}{\partial x^{\nu}}(\eta_{\alpha\beta})$$

$$= \eta_{\mu\nu} + h_{\mu\nu} - \frac{\partial \xi_{\alpha}}{\partial x^{\mu}} - \frac{\partial \xi_{\beta}}{\partial x^{\nu}} = \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu}$$

$$h_{\mu'\nu'} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \qquad h' \approx h - 2\xi_{\mu}{}^{,\mu}$$

$(h'_{\mu\nu})$

$$\left(\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \qquad \bar{h} = \bar{h}^{\alpha}{}_{\alpha} = -h(= -h^{\alpha}{}_{\alpha})\right)$$

$$\bar{h}_{\mu'\nu'} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi_{\beta}{}^{,\beta}$$

To Satisfy Lorentz Gauge Condition $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0, \bar{h}'^{\mu\alpha}{}_{,\alpha} = 0$

$$\bar{h}'^{\mu\alpha}{}_{,\alpha} = \bar{h}^{\mu\alpha}{}_{,\alpha} - \xi^{\mu,\alpha}{}_{\alpha} + \underline{\left(\eta^{\mu\alpha}\xi_{\beta}{}^{,\beta}{}_{\alpha} - \xi^{\alpha,\mu}{}_{\alpha}\right)}_{0}$$

Einstein Equation Linearization

(2-4) Einstein Equation with Gauge Condition

To Satisfy Lorentz Gauge Condition $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0, \overline{h'}^{\mu\alpha}{}_{,\alpha} = 0$

$$\overline{h'}^{\mu\alpha}{}_{,\alpha} = \underbrace{\bar{h}^{\mu\alpha}{}_{,\alpha} - \xi^{\mu,\alpha}{}_{\alpha}}_{0} + \underbrace{\left(\eta^{\mu\alpha}\xi_{\beta}{}^{,\beta}{}_{\alpha} - \xi^{\alpha,\mu}{}_{\alpha}\right)}_{0}$$

$$\therefore\ \bar{h}^{\mu\alpha}{}_{,\alpha} = \xi^{\mu,\alpha}{}_{\alpha} = \eta^{\alpha\beta}\xi^{\mu}{}_{,\beta\alpha} = \left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\xi^{\mu} \equiv \Box\xi^{\mu}$$

$$\therefore\ G_{\mu\nu} = \frac{1}{2}\left(-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} \underbrace{- \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{\nu} + \bar{h}_{\nu\alpha}{}^{,\alpha}{}_{\mu}\right)}_{0}$$

$$\therefore\ G_{\mu\nu} = -\frac{1}{2}\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = -\frac{1}{2}\eta^{\alpha\beta}\bar{h}_{\mu\nu,\alpha\beta} = -\Box\bar{h}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein Equation Linearization

(2-6) Einstein equation is recognized as the following wave equation.

$$G_{\mu\nu} = \frac{1}{2}\left(-\bar{h}_{\mu\nu,\alpha}{}^{\alpha}\right) = -\Box\,\bar{h}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \bar{h}_{\mu\nu}^{GP}(t,\vec{x}) + \bar{h}_{\mu\nu}^{GW}(t,\vec{x})$$

$\bar{h}_{\mu\nu}^{GW}(t,\vec{x})$ **is the gravitational waves !!**

(3) Gravitational waves in the vacuum.

$$\Box\,\bar{h}_{\mu\nu} = 0, \left(T_{\mu\nu} = 0\right)$$

The solution is a plane wave function

$$\bar{h}_{\mu\nu} = a_{\mu\nu}\, e^{(i\, k_\alpha\, x^\alpha)} \qquad \left(a_{\mu\nu} = a_{\nu\mu}\right)$$

$a_{\mu\nu}$: Polarization Tensor (Complex, Take real part)

k_α : Wave number, Null Vector

Gravitational Waves in TT Gauge

(4-1) Transverse-Traceless (TT) Gauge.

Apply $\bar{h}_{\mu\nu} = a_{\mu\nu}\, e^{(i\, k_\alpha\, x^\alpha)}$

into $\Box \bar{h}_{\mu\nu} = 0 \Rightarrow k_\alpha k^\alpha = 0$ Propagation at the speed of light

into $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0 \Rightarrow a_{\mu\alpha} k^\alpha = 0$ Propagation and wave vector are orthogonal.

$h_{\mu\nu}$ is symmetric matric $\Rightarrow a_{\mu\nu} = a_{\nu\mu}$

10 degrees of freedom is expected to be 6, because the Lorentz gauge condition supplies independent 4 conditions.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Gravitational Waves in TT Gauge

(4-1) Transverse-Traceless (TT) Gauge.

Apply $\bar{h}_{\mu\nu} = a_{\mu\nu}\, e^{(i\, k_\alpha\, x^\alpha)}$

into $\Box\, \bar{h}_{\mu\nu} = 0$ ➡ $k_\alpha k^\alpha = 0$ Propagation at the speed of light

into $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ ➡ $a_{\mu\alpha} k^\alpha = 0$ Propagation and wave vector are orthogonal.

$h_{\mu\nu}$ is symmetric matric ➡ $a_{\mu\nu} = a_{\nu\mu}$

However, Lorentz gauge are not fully 6 the coordinates.
In fact, if we perform another infinitesimal coordinate transformation ($\sigma^{\mu\prime}$ to $\sigma^{\mu\prime\prime}(h' \rightarrow h'')$), keeping $\Box\sigma^\mu = 0$ and Lorentz condition.

So we can reduce another 4 degrees of freedom.
Consequently the remained degrees of freedom is 2 !

Gravitational Waves in TT Gauge

(4-2) Transverse-Traceless (TT) Gauge.

Apply $\xi^{\mu} \rightarrow \xi^{\mu} + \sigma^{\mu}, \left(\sigma^{\mu} \ll 1, \sigma^{\mu}{}_{,\nu} = O(h)\right)$ **In the previous Eqs.**

$$\overline{h''}_{\mu\nu} = \overline{h}'_{\mu\nu} - \sigma_{\mu,\nu} - \sigma_{\nu,\mu} + \eta_{\mu\nu}\sigma_{\beta}{}^{,\beta}$$

$$\overline{h''}^{\mu\alpha}{}_{,\alpha} = \overline{h}'^{\mu\alpha}{}_{,\alpha} - \sigma^{\mu,\alpha}{}_{\alpha}$$

As long as $\Box\sigma^{\mu} = 0$ is satisfied, the dof of σ^{μ} has nothing to do with the Gauge Condition.

Gravitational Waves in TT Gauge

(4-2) Transverse-Traceless (TT) Gauge.

Set $\sigma^{\mu} = -ib^{\mu} e^{(i\,k_{\alpha}\,x^{\alpha})}$ **then,** $\Box\sigma^{\mu} = 0$ **is satisfied.**

Apply $\xi^{\mu} \rightarrow \xi^{\mu} + \sigma^{\mu}, \left(\sigma^{\mu} \ll 1, \sigma^{\mu}{}_{,\nu} = O(h)\right)$

$$\overline{h''}_{\mu\nu} = \overline{h}'_{\mu\nu} - \sigma_{\mu,\nu} - \sigma_{\nu,\mu} + \eta_{\mu\nu}\sigma_{\beta}{}^{,\beta}$$

$$= \underline{a_{\mu\nu}\, e^{(i\,k_{\alpha}\,x^{\alpha})} - \left(b_{\mu}k_{\nu} + b_{\nu}k_{\mu} - \eta_{\mu\nu}b^{\beta}k_{\beta}\right)e^{(i\,k_{\alpha}\,x^{\alpha})}}$$

$$-\xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi_{\beta}{}^{,\beta}$$

$$\Downarrow$$

$$a_{\mu\nu} \rightarrow a_{\mu\nu} - \left(b_{\mu}k_{\nu} + b_{\nu}k_{\mu} - \eta_{\mu\nu}b^{\beta}k_{\beta}\right)$$

$a_{\mu\nu}$ **replacement shows 4 dof corresponds to gauge freedom among 6 dof.**
Consequently, the rest dof is only 2 !!

Gravitational Waves in TT Gauge

(4-3) Transverse-Traceless (TT) Gauge.

Using the degree of freedom of b^{μ}, we can select TT gauge satisfying with the following conditions,

$a_{0\mu} = 0 \Longrightarrow a_{ij}k^{j} = 0$ **Transverse Waves**

$a^{j}{}_{j}\left(= \eta^{ji}a_{ij}\right) = 0$ **Traceless Wave Amplitude**

ν **: 0,1,2,3**
i, j **: 1,2,3**

In this TT gauge, $\bar{h}_{\mu\nu} = h_{\mu\nu}$

TT gauge conditions can be written by considering k_{μ}

$$\bar{h}^{TT}_{\mu 0} = 0, \qquad \bar{h}^{TT,j}_{ij} = 0, \qquad \bar{h}^{TT}\left(= \eta^{ji}h^{TT}_{ij}\right) = 0$$

Gravitational Waves in TT Gauge

(4-4) Transverse-Traceless (TT) Gauge.

Assuming Cartesian coordinates, GWs travelling along z (x^3) axis, and $k_\alpha = (-k, 0{,}0, k)$

$$h_{\mu\nu}^{TT} = \bar{h}_{\mu\nu}^{TT} = a_{\mu\nu}^{TT}\, e^{i\,(-\omega t + kz)}$$

$$h_{\mu 0}^{TT} = 0, h_{ij}^{TT,j} = 0, \eta^{ji} h_{ij}^{TT} = 0 \Longrightarrow a_{\mu 0} = 0, a_{i3} = 0, a_{11} + a_{22} = 0$$

TT gauge conditions

Note : $a_{\mu\nu} = a_{\nu\mu}$

$$\therefore\ a_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Independent parameters are only two, h_+ **and** $h_\times$

GW Effects on Free Mass

(5) Ripple of Minkowski Space

Line element

$$dl^2 = \left(\delta_{ij} + h_{ij}^{TT}\right)dx^i dx^j$$

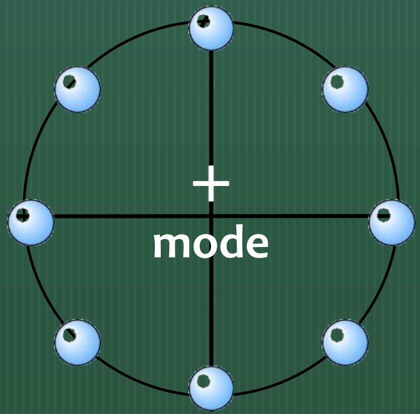
$$= \{1 + h_+ \cos\omega(t-z)\}dx^2 + \{1 - h_+ \cos\omega(t-z)\}dy^2$$
$$+ 2h_\times \cos\omega(t-z)\, dxdy + dz^2$$

$$\equiv \delta_{ij} dx'^i dx'^j$$

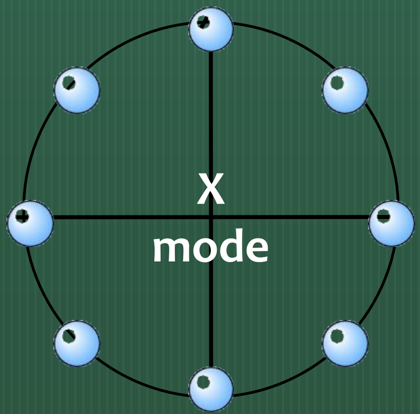
Where

$$dx' = \left(1 + \frac{h_+ \cos\omega(t-z)}{2}\right)dx + \frac{h_\times \cos\omega(t-z)}{2}dy$$

$$dy' = \left(1 - \frac{h_+ \cos\omega(t-z)}{2}\right)dy + \frac{h_\times \cos\omega(t-z)}{2}dx$$



Put free mass at
$(dx, dy) = (\cos\theta, \sin\theta)$



Geodesics Equation to Detect GWs

To detect GWs effect on free mass, we should know the differential motion of two free ($x^\mu, x^\mu + \xi^\mu$) falling objects with external force $F^\mu(x)$.

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}(x)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = F^\mu(x)$$

$$\frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}(x+\xi)\frac{d(x^\alpha + \xi^\alpha)}{d\tau}\frac{d\left(x^\beta + \xi^\beta\right)}{d\tau} = F^\mu(x+\xi)$$

Take difference between these equations, and take 1 dim of ξ^μ

$$\frac{d^2 \xi^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta,\gamma}(x)\xi^\gamma\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + 2\Gamma^\mu{}_{\alpha\beta}(x)\frac{d\xi^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = F^\mu{}_{,\nu}\xi^\nu$$

$$\Longrightarrow \frac{D^2 \xi^\mu}{D\tau^2} + R^\mu{}_{\alpha\beta\gamma}(x)\xi^\beta\frac{dx^\alpha}{d\tau}\frac{dx^\gamma}{d\tau} = F^\mu{}_{,\nu}\xi^\nu$$

Here we use $\frac{D\xi^\mu}{D\tau} = \frac{d\xi^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta}\xi^\alpha\frac{dx^\beta}{d\tau}$

Geodesics Equation to Detect GWs

Assuming GW travelling along Z-axis, $d\tau \approx dt, \frac{dx^0}{d\tau} \approx 1, \frac{dx^i}{d\tau} \approx 0$

$$\frac{D^2\xi^\mu}{D\tau^2} + R^\mu{}_{\alpha\beta\gamma}(x)\xi^\beta \frac{dx^\alpha}{d\tau}\frac{dx^\gamma}{d\tau} = f^\mu \Longrightarrow \frac{d^2\xi^j}{dt^2} + R^j{}_{0k0}\xi^k = f^\mu \qquad R^\nu{}_{000} = 0$$

Where, we should note in TT gauge, $\Gamma^\nu{}_0 \approx 0, \Gamma^\nu{}_{0k} \approx \frac{1}{2} h^{TT\,j}{}_{k,0}$

$$R^j{}_{0k0} \approx -\frac{1}{2}\frac{\partial h^{TT\,j}{}_k}{\partial t^2}$$ **Is obtained.**

Consequently, $$\frac{d^2\xi^j}{dt^2} + R^j{}_{0k0}\xi^k = f^\mu \Longrightarrow \frac{d^2\xi^j}{dt^2} = \frac{1}{2}\frac{\partial h^{TT\,j}{}_k}{\partial t^2}\xi^k + f^\mu$$

Gravitational Wave Force

Geodesics Equation to Detect GWs

We can use the following approximations,

$$\frac{d^2\xi^3}{dt^2} = 0, \qquad \frac{d^2\xi^1}{dt^2} = \frac{1}{2}\frac{\partial h_+}{\partial\tau^2}\xi^1 \qquad \frac{d^2\xi^2}{dt^2} = -\frac{1}{2}\frac{\partial h_+}{\partial\tau^2}\xi^2$$

Integral $$\frac{d\xi^1}{\xi^1} = -\frac{d\xi^2}{\xi^2} = \frac{h_+}{2}$$

These results show that GW effect is …

(1) Relative distance change (Strain)

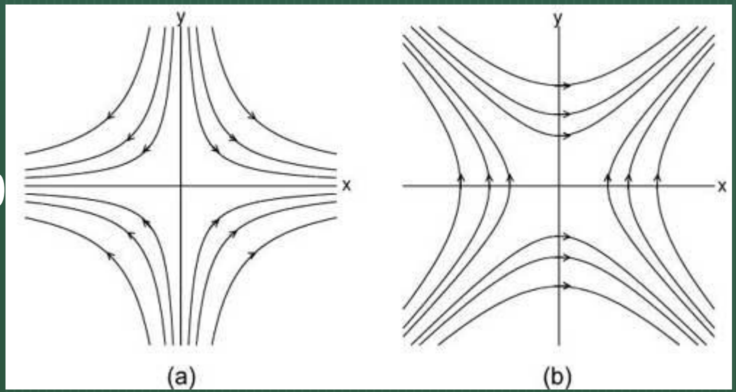
(2) Differential Motion → Michelson Interferometer

Line of GW Force $\left(x(m), y(m)\right)$**, assuming a coupling constant** K

$$\frac{dx}{dm} = K\frac{1}{2}\frac{\partial h_+}{\partial\tau^2}x \qquad \frac{dy}{dm} = -K\frac{1}{2}\frac{\partial h_+}{\partial\tau^2}y$$

Remove m

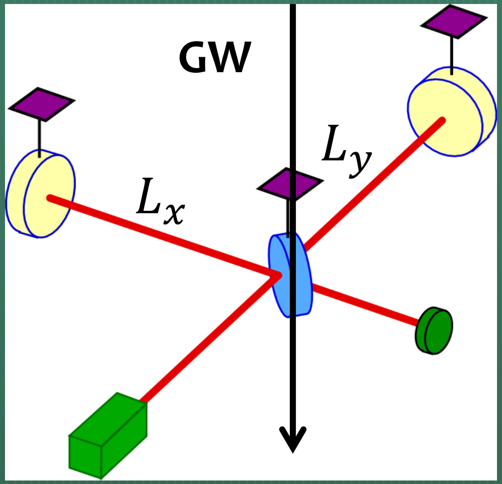
$$\frac{dx}{x} + \frac{dy}{y} = 0$$



Michelson Interferometer

Assuming GW travelling along Z axis,

$$ds^2 = -c^2dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + dz^2$$



For the light travelling x axis

$$ds^2 = -c^2dt^2 + (1+h_+)dx^2 \Longrightarrow \frac{dx}{dt} = \pm c\frac{1}{\sqrt{1+h_+}}$$

This means that the superficial speed of light changes.
The time to go and back the Michelson Interferometer arm is,

$$\int_{t-t_x}^{t}\frac{1}{\sqrt{1+h_+(t')}}dt' = \frac{1}{c}\left(\int_0^{L_x}dx - \int_{L_x}^{0}dx\right) = \frac{2L_x}{c}$$

$$t_x = \frac{2L_x}{c} + \frac{1}{2}\int_{t-L_x/c}^{t}h_+(t')dt' \qquad t_y = \frac{2L_y}{c} - \frac{1}{2}\int_{t-L_y/c}^{t}h_+(t')dt'$$

The second term is the effect of GW

Michelson Interferometer

So, the differential motion due to GW (ϕ_{GW}) is,

$$\boldsymbol{\phi_{GW}(t)} = \Omega \int_{t-L/c}^{t} h_+(t')dt'$$

$$L_x \approx L_y = L$$

Ω: Angular Frequency of Light

Assuming, $h_+(t) = h_0 e^{\left(-i\,\omega\left(t-\frac{L}{c}\right)\right)}$,

ω: Angular Frequency of GW

$$\boldsymbol{\phi_{GW}(t)} = \frac{\boldsymbol{2\Omega h_0}}{\boldsymbol{\omega}} \sin\left(\frac{\omega L}{c}\right) e^{\left(-i\,\omega\left(t-\frac{L}{c}\right)\right)}$$

$$\boldsymbol{\phi_{GW}(t)} = \frac{\boldsymbol{2\Omega h_0 L}}{\boldsymbol{c}} \text{ if} \frac{\omega L}{c} \ll 1$$

If the turnaround time is same with a half period of GW, the phase difference is max. If longer, the effect of GW will be cancelled.

Gravitational Wave Radiation

Assuming

(1) the gravity potential is small,

(2) the speed in side the GW sources is much less than the speed of light.

Then, the GWs are derived from the Quadrupole Formula

$$h_{ij}^{TT}(t,x) = \frac{2G}{r\,c^4}\ddot{Q}_{ij}^{TT}\left(t - \frac{r}{c}\right)$$

$$Q_{ij} = \int d^3x\rho(t,x)\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right) \Longrightarrow \ddot{Q}_{ij}^{TT} = \Lambda_{ij,kl}\,Q_{kl}$$

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

$$P_{ij} = \delta_{ij} - n_i n_j \qquad n_i = \frac{x^i}{r}$$

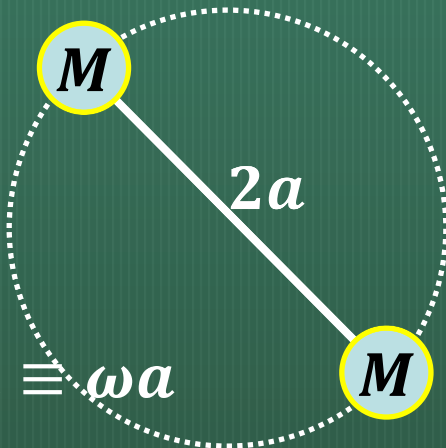
(Artificial) Gravitational Wave Radiation

GW Luminosity

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \langle \dot{h}_{ij}^{TT} \, \dot{h}^{ij\,TT} \rangle = \frac{G}{5c^5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle$$

GW Angular Momentum Luminosity

$$\frac{dJ^i}{dt} = \frac{G}{5c^5} \varepsilon^{ikl} \langle \ddot{Q}_{ka} \dddot{Q}^{ka} \rangle$$



$$\boldsymbol{v = 2\pi f a \equiv \omega a}$$

Can We generate GWs ??

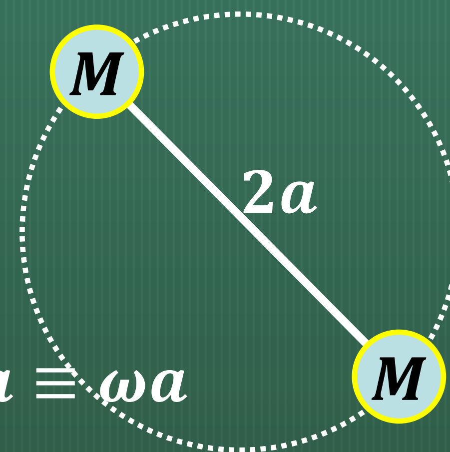
$$\frac{dE}{dt} = \frac{128G}{5c^5} M^2 a^4 \omega^6 = 5 \times 10^{-33} \left(\frac{M}{100[\mathrm{kg}]}\right)^2 \left(\frac{2a}{10[\mathrm{m}]}\right)^4 \left(\frac{f}{10[\mathrm{Hz}]}\right)^6 [\mathrm{W}]$$

(Artificial) Gravitational Wave Radiation

<u>Expected GW strain amplitude. :</u>

$$h_{ij}^{TT} = \frac{2G}{rc^4}\ddot{Q}_{ij}^{TT} \sim \frac{8G}{rc^4}Ma^2\omega^2$$

The smaller r results in the larger h_{ij}^{TT} .
However, we should apart far from the GW source than the wavelength of GWs. Otherwise, we cannot treat GWs as waves.

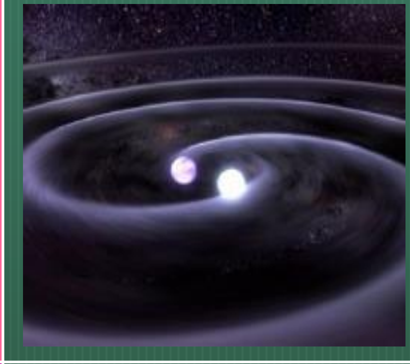
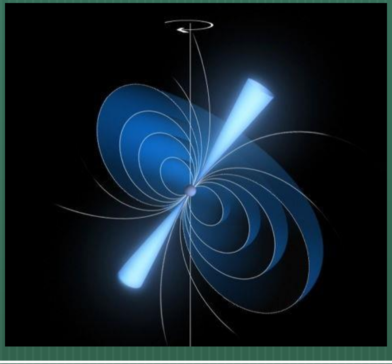
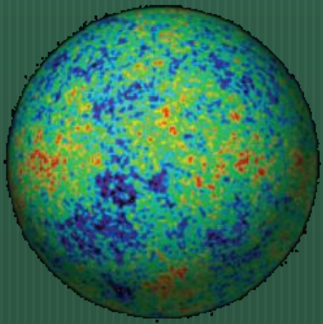


$$v = 2\pi f a \equiv \omega a$$

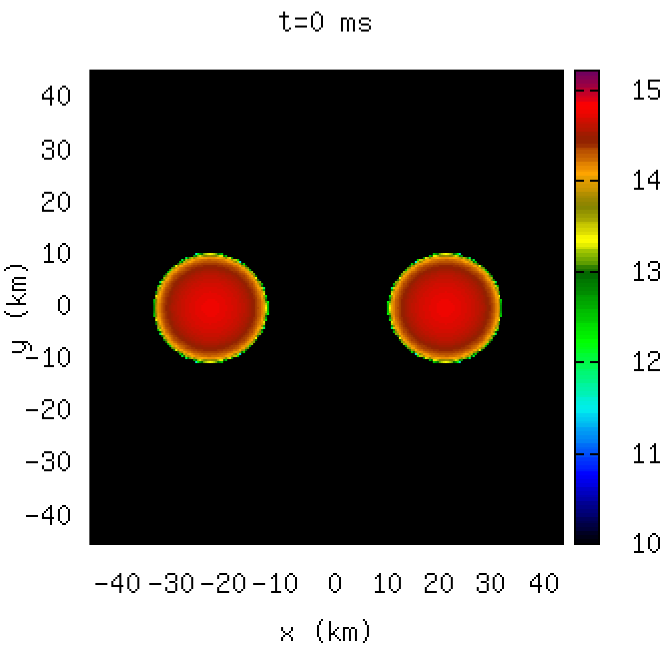
Assuming 10Hz angular frequency, the wavelength of GW becomes 15000km ~ the diameter of the Earth.

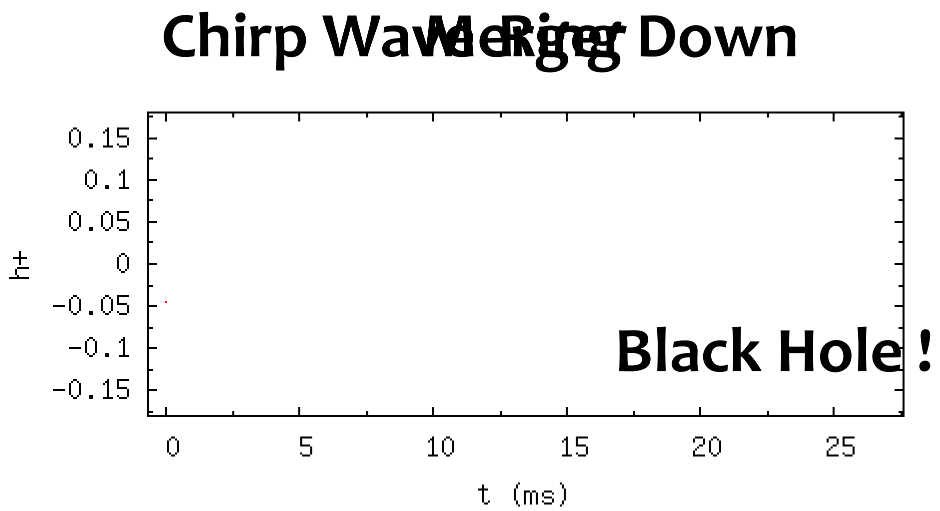
$$h < \frac{8G}{\pi c^5}Ma^2\omega^3 = 5 \times 10^{-45} \left(\frac{M}{100[\mathrm{kg}]}\right)^2 \left(\frac{2a}{10[\mathrm{m}]}\right)^4 \left(\frac{f}{10[\mathrm{Hz}]}\right)^3 [\mathrm{W}]$$

Gravitational Wave Sources and Wave Forms

	Short Duration Time (GW Events)	Long Duration Time (Stationary GW)
Known Wave Form	**Chirp Wave**  CBC NS-NS, NS-BH, BH-BH	**Continuous Wave**  Pulsar Neutron Star
Unknown Wave Form	**Burst Like Wave**  Supernovae Soft Gamma Ray Pulsar Glitch GRB	**Random Wave**  Stochastic Promodial Cosmic String Accumulation

Compact Binary Coalescences

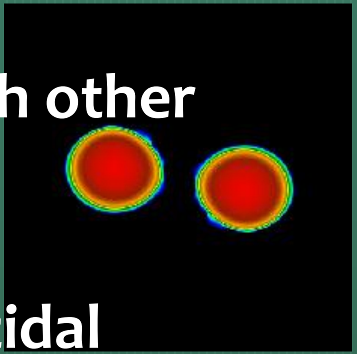




Dr. Hotokezaka (Kyoto U)
http://www2-tap.scphys.kyoto-u.ac.jp/~hotoke/anime/rho_gw_H4_14_60.gif

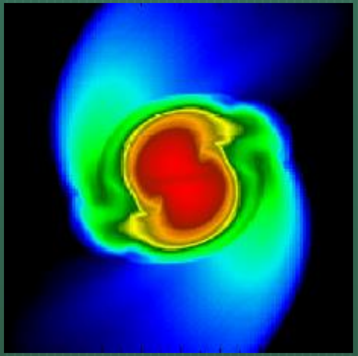
What happens during Compact Binary Coalescences

Inspiral (post-Newtonian)



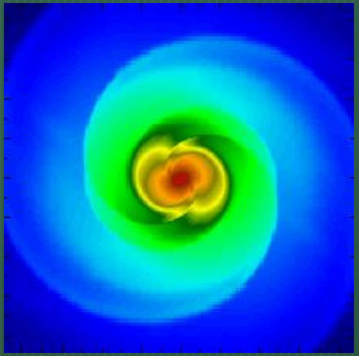
- A pair of NS and/or BHs are approaching with each other because of the energy loss due to GWs radiation.
- The orbit is Quasi-stable circular orbit.
- They can be treated with a point mass free from tidal deformation just before the merger.

Merger (Numerical Relativity)



- Innermost stable circular orbit (ISCO).
- Compact stars deformation, mixing.

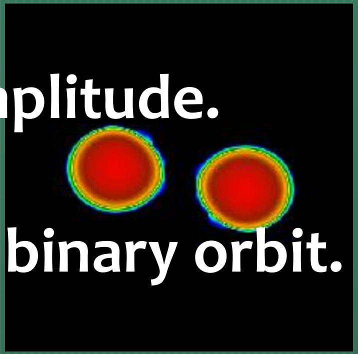
Ringdown



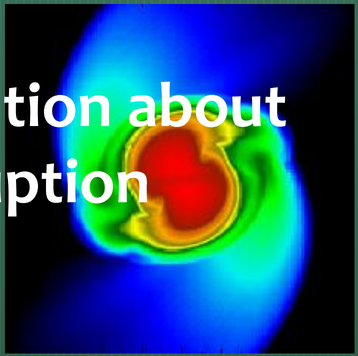
- Hyper massive NS and transition to BH formation

Physics from GWs during the Each Phase of Compact Binary Coalescences

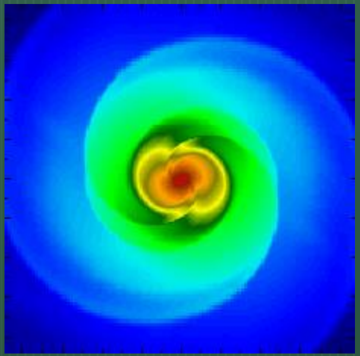
- Inspiral (post-Newtonian)
  - Frequency development → mass and absolute amplitude.
  - Measured amplitude → distance from the Earth.
  - Measured polarization → inclination angle of the binary orbit.



- Merger (Numerical Relativity)
  - Depends on many conditions → complex information about stars, e.g. radius, viscosity, EOS, tidal effect (disruption deformation).



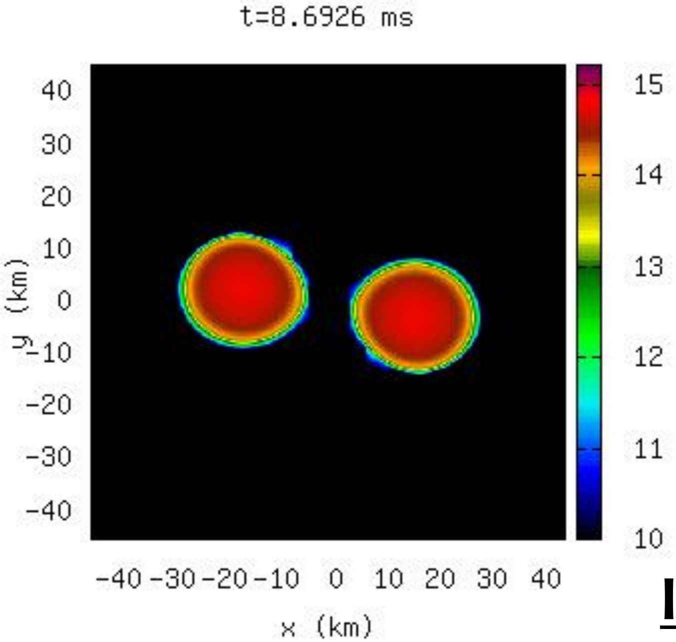
- Ringdown
  - BH quasi-normal mode.
  - Frequency → mass.
  - Decay time → Spin (Karr Parameter)

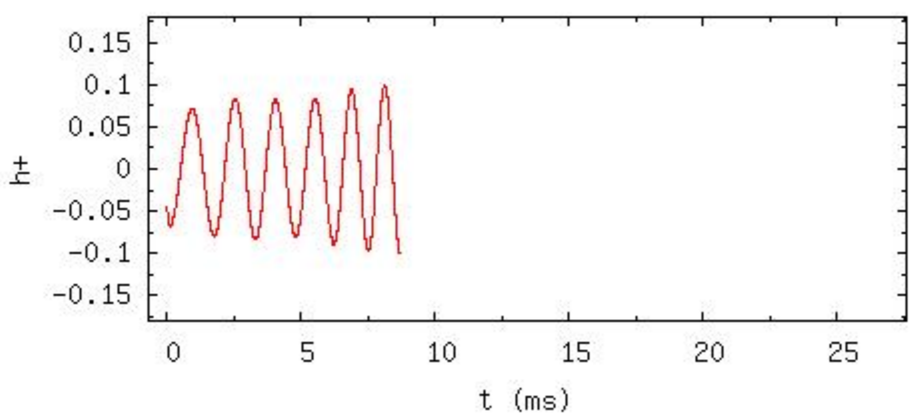


Rotation (Rot.) and Inspiral (Ins.) Phase

Rotation →

- **NS, BH Mass, Period, Orbital Plane inclination, etc**
- **Low frequency GW radiation and the decrease of revolution period.**





Inspiral →

- **Higher frequency GWs radiation (Main Target of GW Detectors)**

Rot. and Inspiral Phase (e~0)

$t = 0$ is the coalescence time !

GW strain amplitude

$$h = \frac{4G}{rc^4}\mu a^2\omega^2 \rightarrow \frac{4G^{5/3}}{rc^4} M_{ch}^{5/3}\omega^{2/3}$$

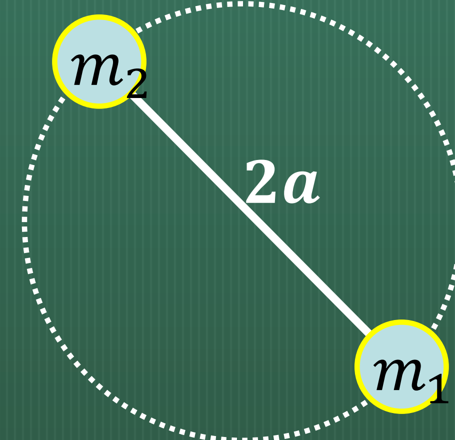
$$h_{ij}^{TT} = \frac{8G}{rc^4} M a^2 \omega^2$$

$$M = m_1 + m_2$$

M_{ch}: Chirp Mass $\equiv \mu^{3/5} M^{2/5}$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Kepler Rule : $GM = a^3\omega^2$



Orbital Radius Decrease

$$\frac{da}{dt} = -\frac{64G^3}{5c^5a^3} M_{ch}^{5/3} M^{4/3}$$

Orbital freq. Decrease

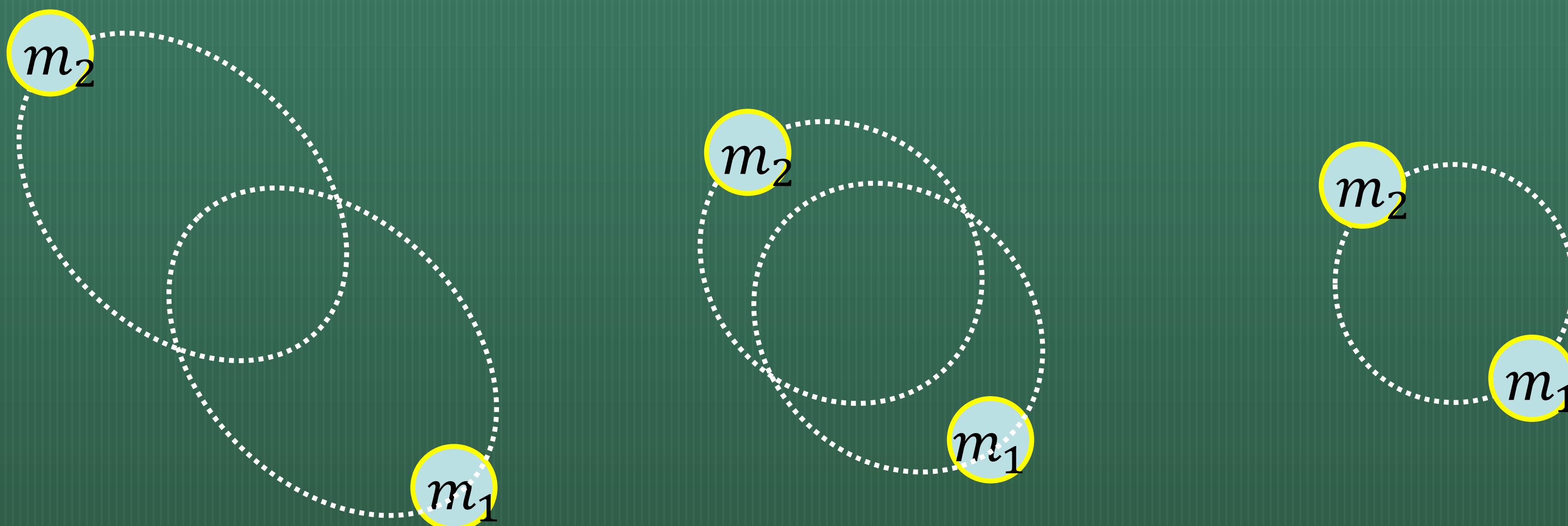
$$\frac{d\omega}{dt} = -\frac{96G^{5/3}}{5c^5} M_{ch}^{5/3}\omega^{11/3}$$

Orbital Period Decrease

$$\frac{dP_b}{dt} = -\frac{96G^{5/3}}{5c^5} M_{ch}^{5/3}\left(\frac{P_b}{2\pi}\right)^{-5/3}$$

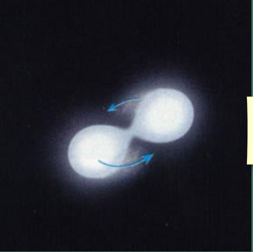
Eccentricity of Rot. &Ins. Phase

How did you expect the eccentricity of the compact binary system will change during radiating GWs ?

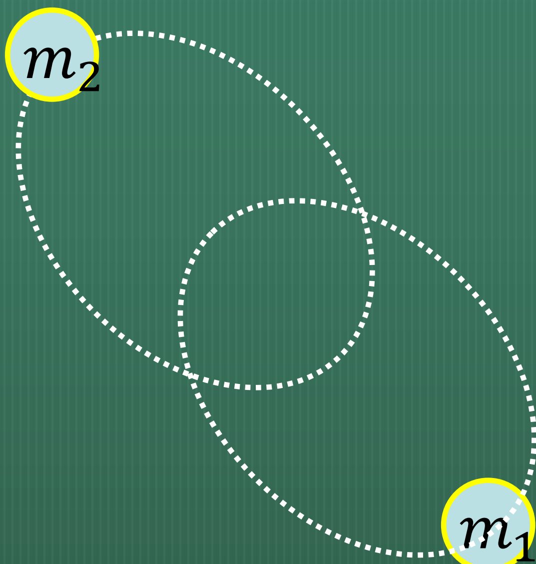


$$\frac{da}{dt} = -\frac{64G^3}{5c^5a^3} M_{ch}^{5/3} M^{4/3} \times f(e) \quad \text{here } f(e) = \left(1 - e^2\right)^{-7/2}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

$$\frac{de}{dt} = -\frac{304G^3}{15c^5a^4} M_{ch}^{5/3} M^{4/3} \times e\left(1 - e^2\right)^{-5/2}\left(1 + \frac{121}{304}e^2\right)$$



Energy and Momentum Change



Energy carried by GW

$$\mathcal{L}_E = \frac{32}{5}\frac{G^4}{c^5 a^5}\mu^2 M^3 \times f(e)$$

$$f(e) = (1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

Momentum carried by GW

$$\mathcal{L}_J = \frac{32}{5}\frac{G^{7/2}}{c^5 a^{7/2}(1-e^2)^2}\mu^2 M^{5/2} \times \left(1+\frac{7}{8}e^2\right)$$

Eccentricity of Rot. & Ins. Phase

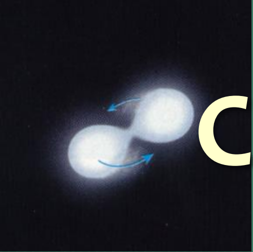
Consequently,

$$e = \left\{ \frac{a}{a_0} g(e_0) \right\}^{\frac{19}{12}} \qquad g(e_0) = \frac{e_0^{12/19}}{1 - e_0^2} \left\{ 1 + \frac{121}{304} e_0^2 \right\}^{\frac{870}{2299}}$$

Before merger, e becomes almost zero, that is circular orbit !!

The shape of chirp signals can be easily predicted from the theoretical calculation including a quadrupole formula and higher order effect of GR (Post Newtonian Effects).

We can apply a Matched Filtering method to effectively find chirp signals in noise !!



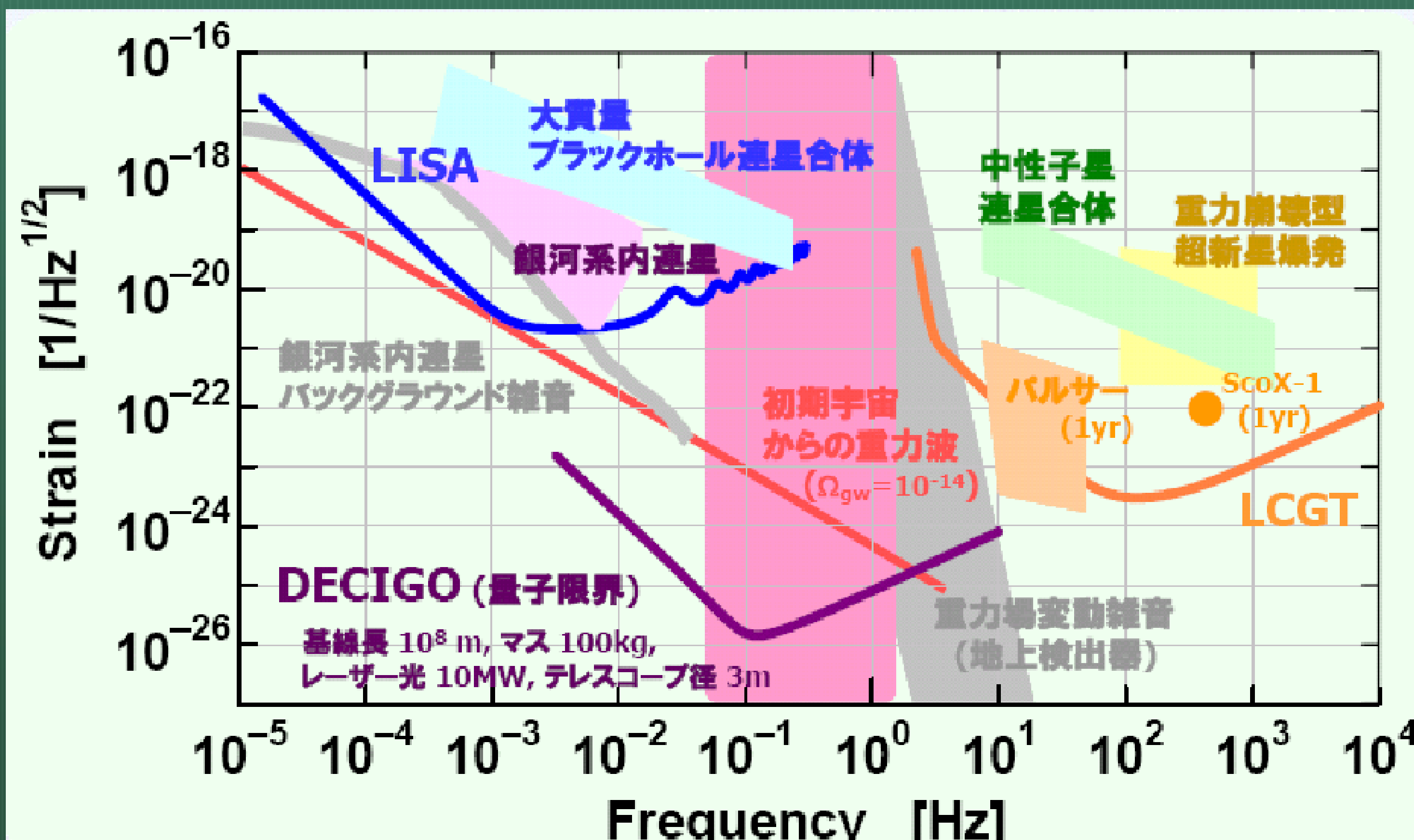
Chirp GW Frequency of Ins. Phase

Frequency depends on the mass of NSs and BHs (Lighter Mass, Higher Frequency).

NS-NS (BH) $f_{GW} = 10 \sim 10^3$ [Hz]

smBH-smBH $f_{GW} = 10^{-4} \sim 10^{-1}$ [Hz]

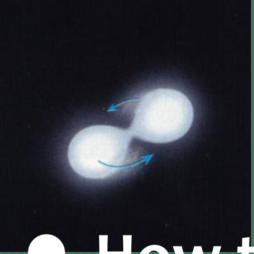
$$f_{GW}(\tau) = \frac{1}{\pi}\left(\frac{5}{256\tau}\right)^{3/8}\left(\frac{GM_{ch}}{c^3}\right)^{-5/8}$$



NS-NS Binary found in/near our Galaxy

- Compact star binary systems are quite rare in our galaxy.
- The following are known systems

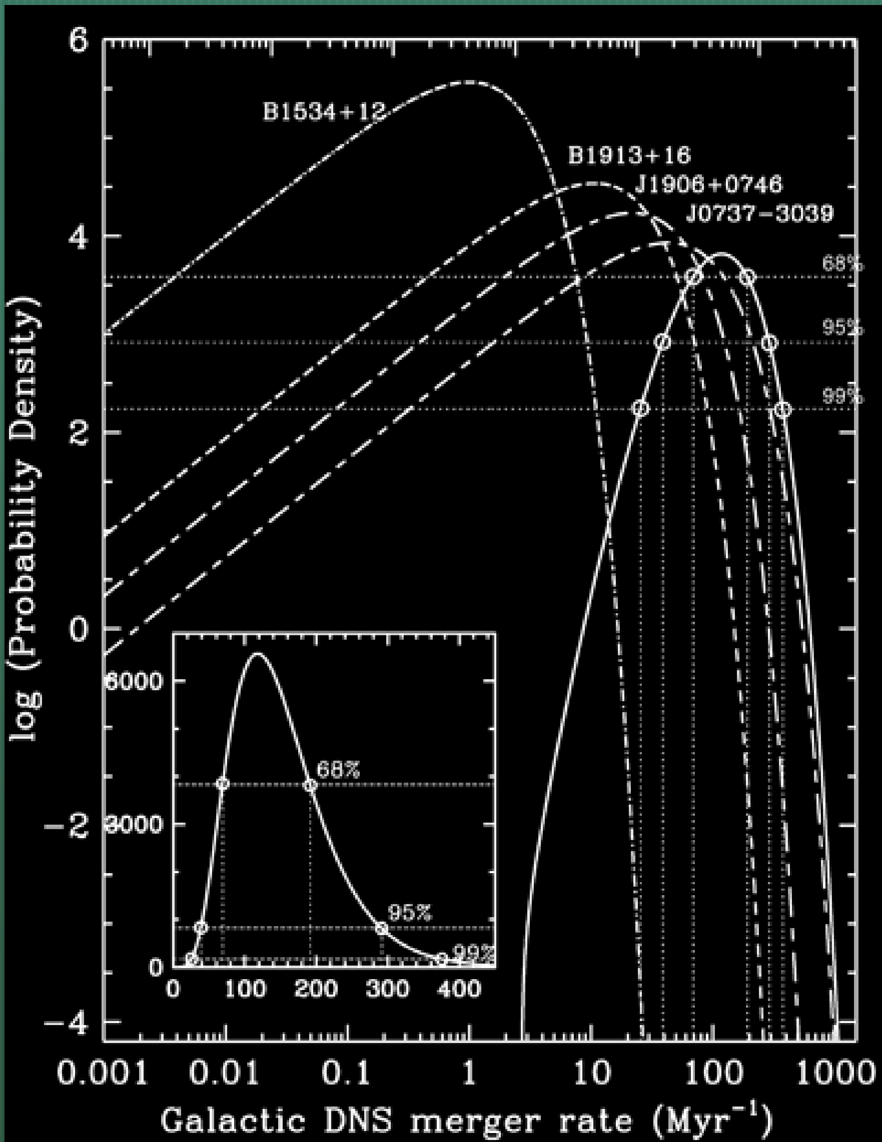
PSR	Merger Time (Myear)	M_1	M_2	note
J0737 – 3039	90	1.34	1.25	Double pulsar
J1906 + 0746	300	1.25	1.37	
J1756 – 2251	1700	1.39	1.18	
B2127 + 11C	220	1.36	1.38	In Cluster
B1913 + 16	330	1.44	1.39	
B1534 + 12	270	1.33	1.35	



Merger Rate of NS-NS Binary

- **How to estimate the merger rate ?**

Kalogera+, ApJ (2004),
Lorimer, LRR (2008), Kim+ (2008)



(1) Pick up only four applicants that are estimated to merge within 13.8 Bys.
(2) Estimate the total numbers of these four kinds of compact binary systems based on their life time, luminosity, probability to be found in the limited observed area and their distribution model in our galaxy,

$$\mathrm{Rate} = 118^{+174}_{-79}\ [\mathrm{events/My/Gal}]$$

- **How often GWDs can detect GWs ?**

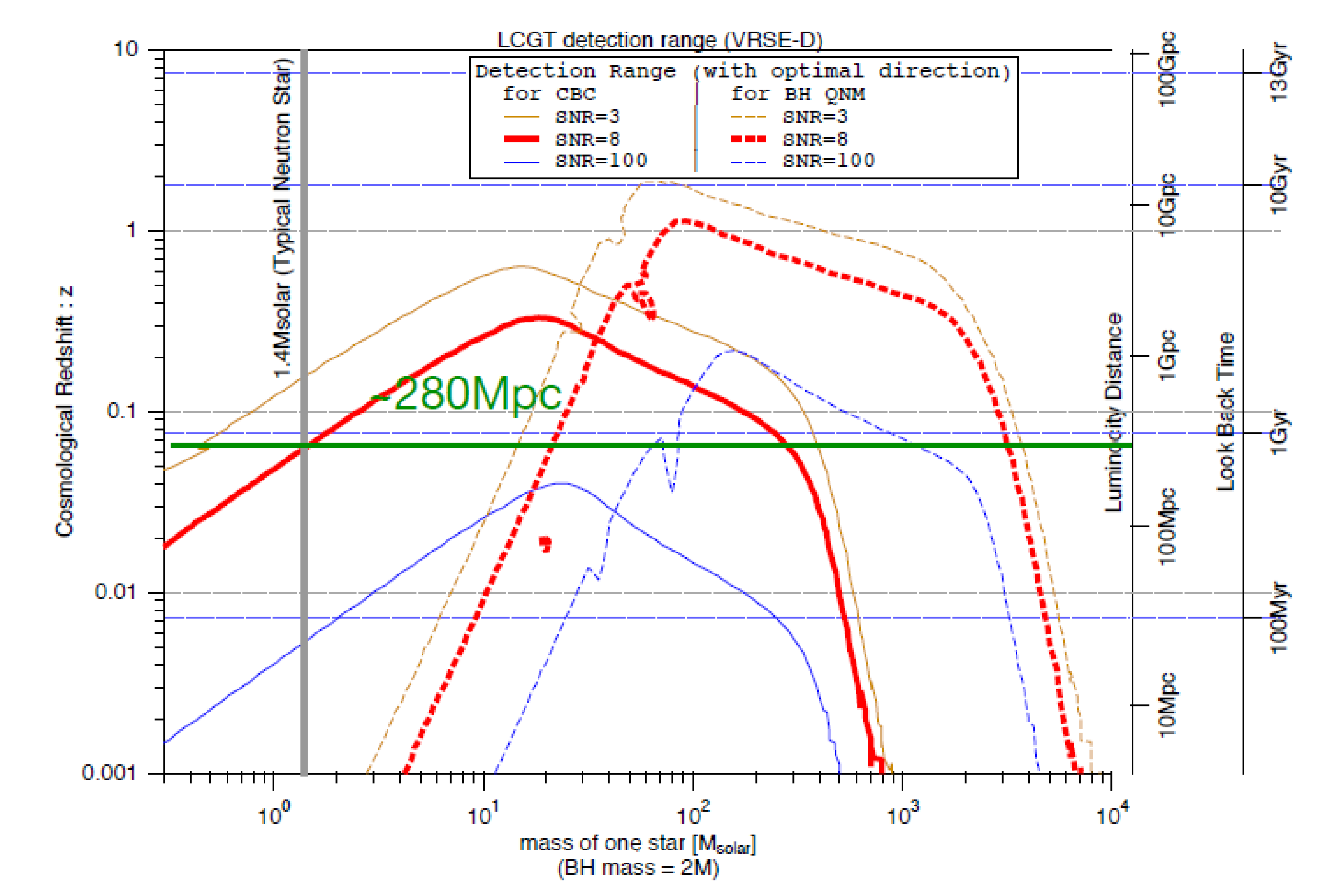
Assuming, galaxy density as

$$\rho_{gal} = 0.012\ [\mathrm{Mpc}^{-3}]$$

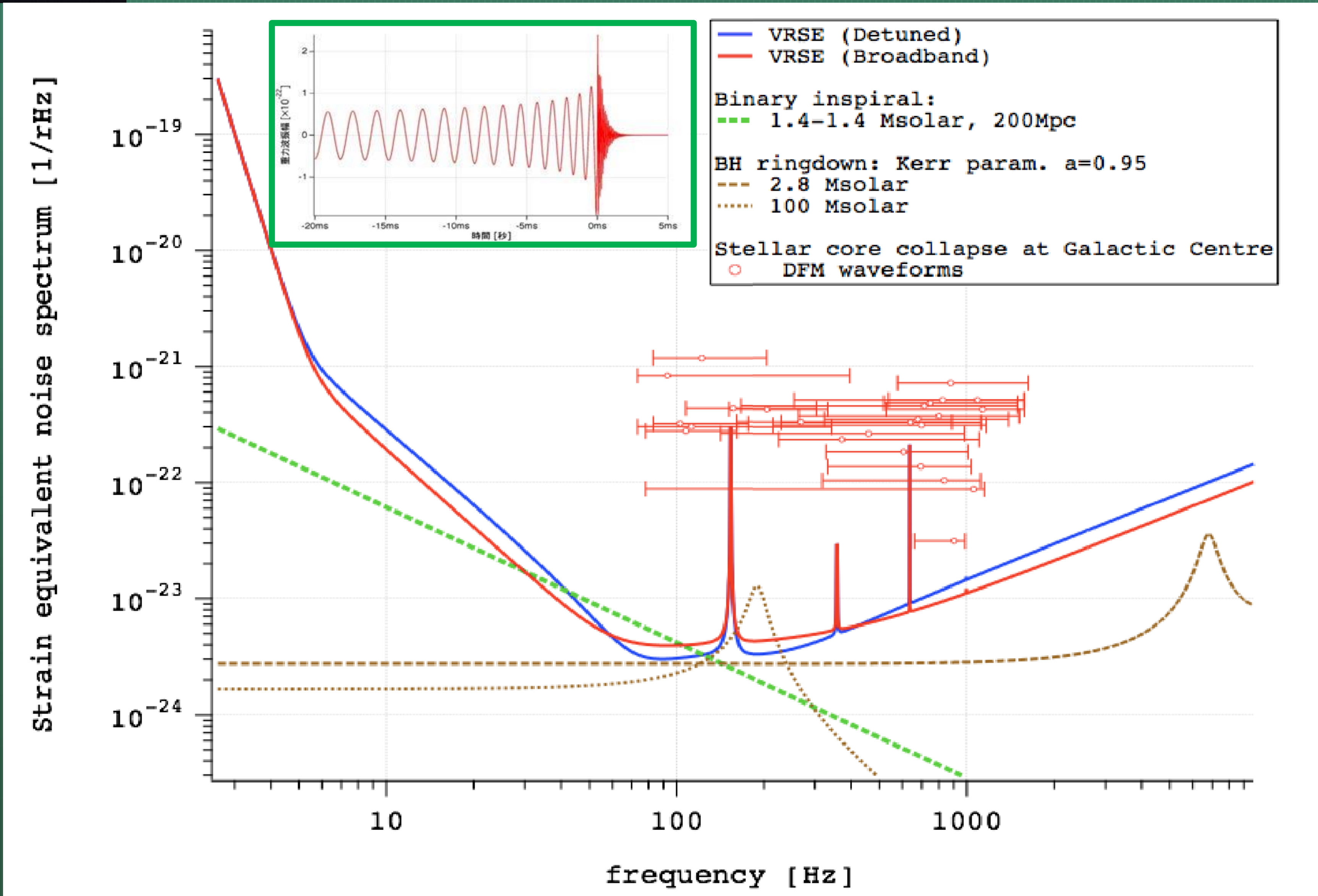
$9.8\ [\mathrm{events/year}]$ assuming 280Mpc detector observable range.

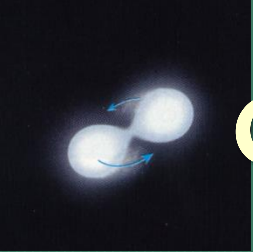
Observable range for NS-NS and BH-BH Binaries (KAGRA)

Observable Range



Chirp Signal Sweep in KAGRA Sensitivity



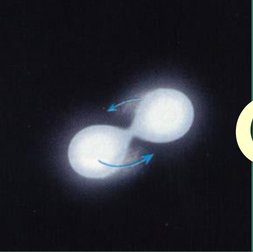


Chirp Signals as Standard Sirens

- In Astronomy, there are several Standard Candles to identify the distance of objects, such as Cepheids variable stars (~20Mpc), Ia supernovae (>20Mpc).

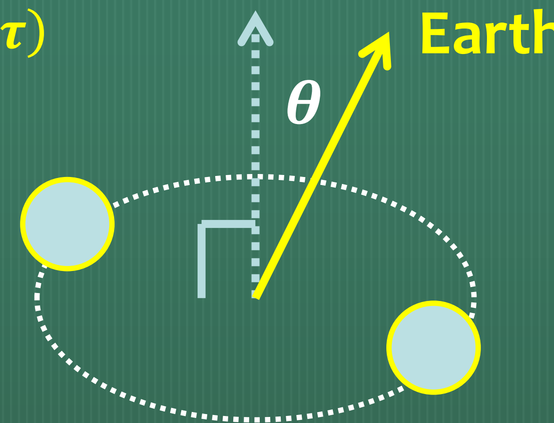
- The distance can be identified by the comparison between known absolute and observed luminosity.

- The Chirp GW signals of NS-NS coalescences are also proposed to use as a "Standard Siren" for the astronomical distance measurement because we can reconstruct the original GW power and shape. (B.Schutz 1986)



Chirp Signals as Standard Sirens

M_{ch} (Chirp Mass) can be derived from $\omega_{GW}(\tau)$ and $\dot{\omega}_{GW}(\tau)$



$$\omega_{GW}(\tau) = 2\left(\frac{5}{256\tau}\right)^{3/8}\left(\frac{GM_{ch}}{c^3}\right)^{-5/8}$$

$$\dot{\omega}_{GW}(\tau) = \frac{48}{5}\left(\frac{GM_{ch}}{2c^3}\right)^{5/3}\omega_{GW}(\tau)^{11/3}$$

r, θ (distance, inclination Angle) can be derived from $h_+(t)$ and $h_\times(t)$

$$h_+(t) = \frac{A}{r}\left(\frac{\omega_{gw(\tau)}}{2c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos\psi(\tau)$$

$$h_\times(t) = \frac{A}{r}\left(\frac{\omega_{gw(\tau)}}{2c}\right)^{2/3}\cos\theta\sin\psi(\tau)$$

$$A = 4\left(\frac{GM_{ch}}{c^2}\right)^{5/3} \qquad \psi(\tau) = \int_{\tau}^{\tau_0}\omega_{g(\tau)}d\tau = -2\left(\frac{5GM_{ch}}{c^3}\right)^{-5/8}\tau^{5/8} + \psi_0$$

Hubble Constant Measurement

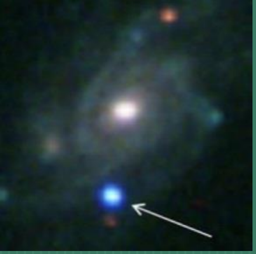
For cosmological distance (z red shift parameter) :

$$\omega_{GW}(\tau) \rightarrow \omega_{GW}(\tau)/(1+z) \quad M_{ch} \rightarrow M_{ch}(1+z) \qquad r \rightarrow D_r(z)$$

Luminosity Distance

- **The actual distance can be derived by measuring "z" independently.**
- **z can be obtained from ex. Its host galaxy red shift. →**
  Distance ladder is no more necessary !!

- **If the short gamma-ray burst's origin is NS-NS coalescence, we can obtain z and other parameters cleanly from GWs of the only NS-NS coalescences.**



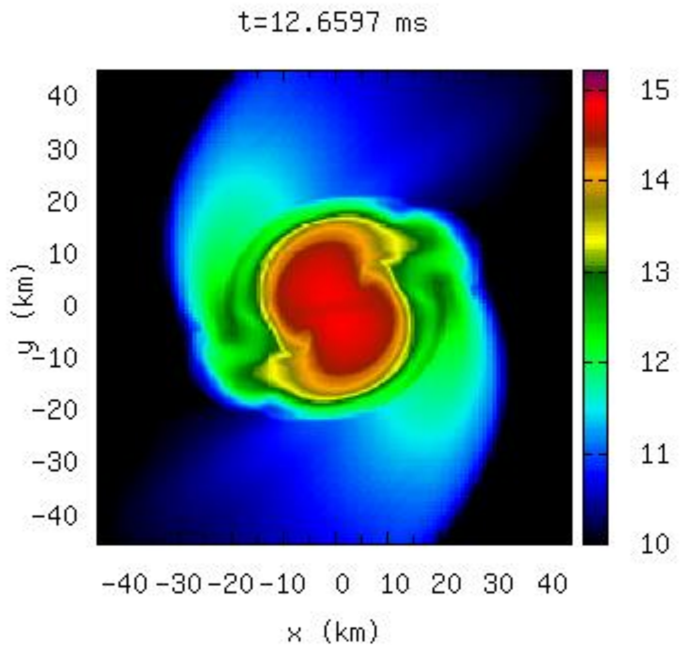
How to Analyze Chirp GWs

- **Matched Filtering** Wave form can be predicted using some parameters, ex mass of NSs.

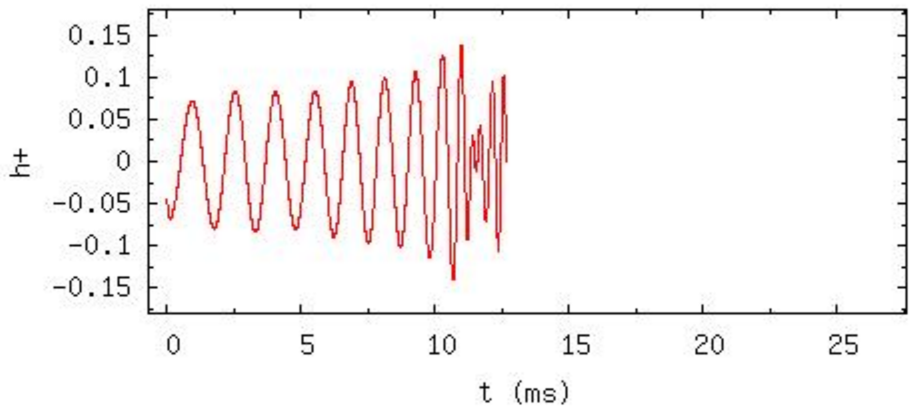
- **Hilbert-Huang Transformation**

Merger Phase

Predicted by a Numerical Simulation using GR.



Complex information about stars, e.g. Radius, Viscosity, EOS, Tidal effect (disruption deformation).

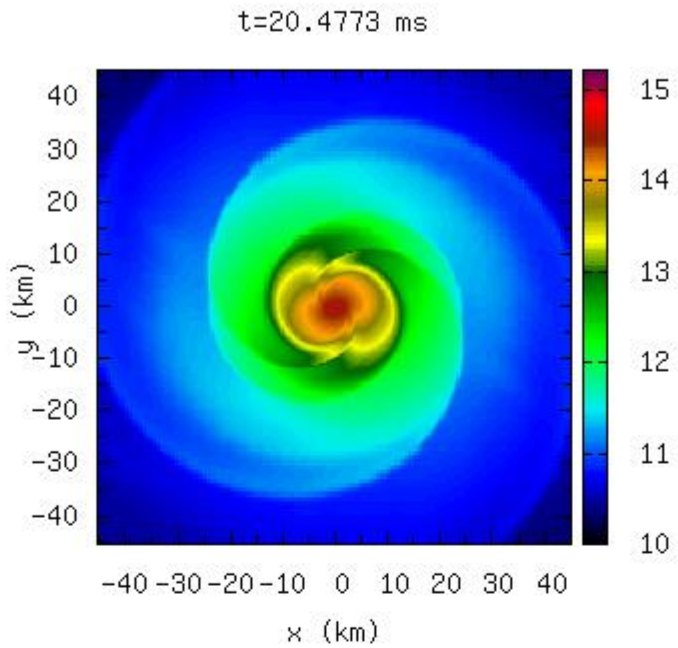


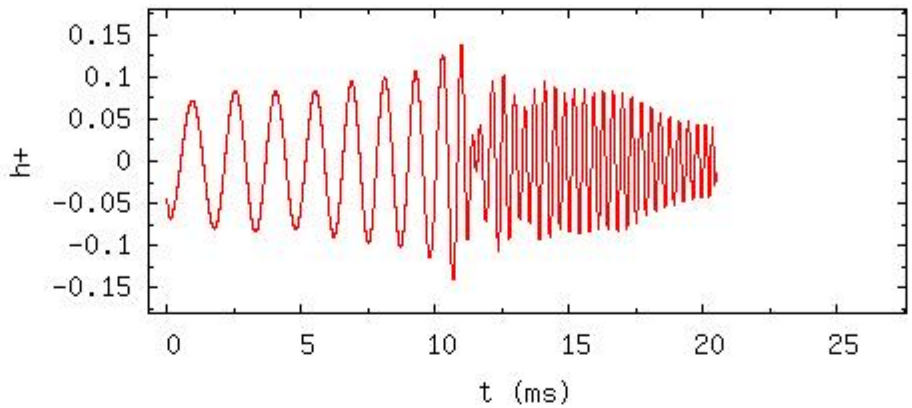
Multi Channel Observation :
Neutrino Radiation and Electro-Magnetic (EM) Waves

Ring down Phase

Black Hole perturbation Methods

Hyper-Massive NS formation. → EOS of NS



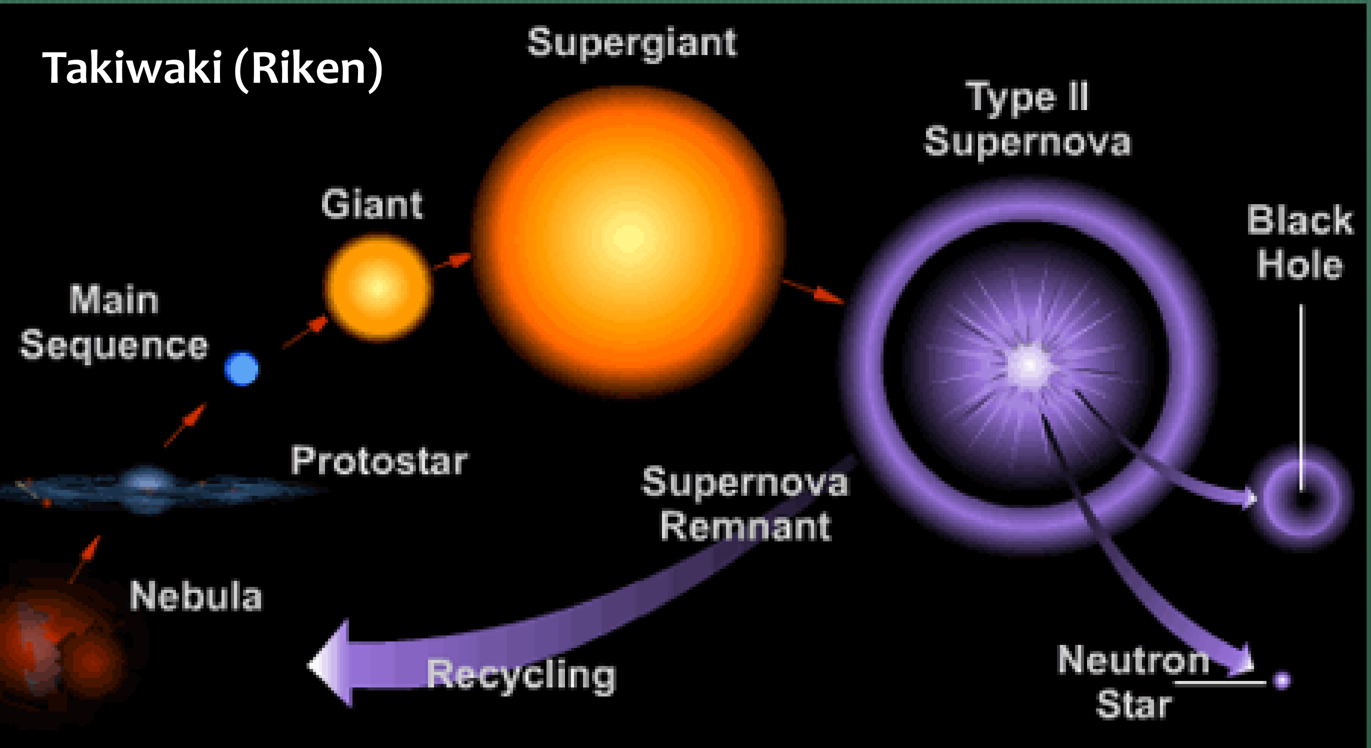


Blackhole Quasi-normal Mode → Mass of BH

Angular Rotation Momentum

GWs from Core Collapse Supernovae

- The core collapse supernovae (Ib, Ic, II type) are applicants of GW radiators.
- A part of non-axis-symmetrical motion energy among supernovae total energy (10^{53} erg) can be converted into GWs. However, we don't know how much it is because the precise mechanism of core collapse supernovae is unknown.
- If the supernovae is aixs-symmetric, no GWs is expected ☹.
- Recently, theoretical supernovae models considering non-axis symmetrical system and 3 dimension simulation succeeded to explode !!



SN Mechanism

1. Electron will be captured in O, Ne, Mg core (30000 km).
2. The core will be Si, Fe (7000 km).
3. Inverse beta decay in Fe produces Neutron (700 km).
4. Hard neuron kick out the outer parts.

Betelgeus is expected to explode in near future

(tomorrow or 100 millions years later ☺)

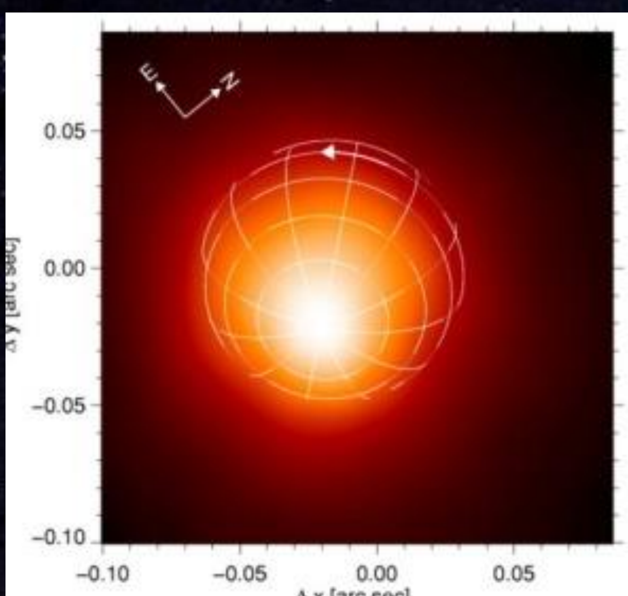
Betelgeus

$M = 7.7 \sim 20\ M_{\odot}$

$R = 950 \sim 1200\ R_{\odot}$

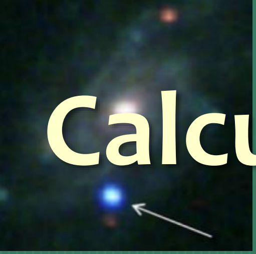
$D = 197 \pm 45\ \text{pc}$

Rigel



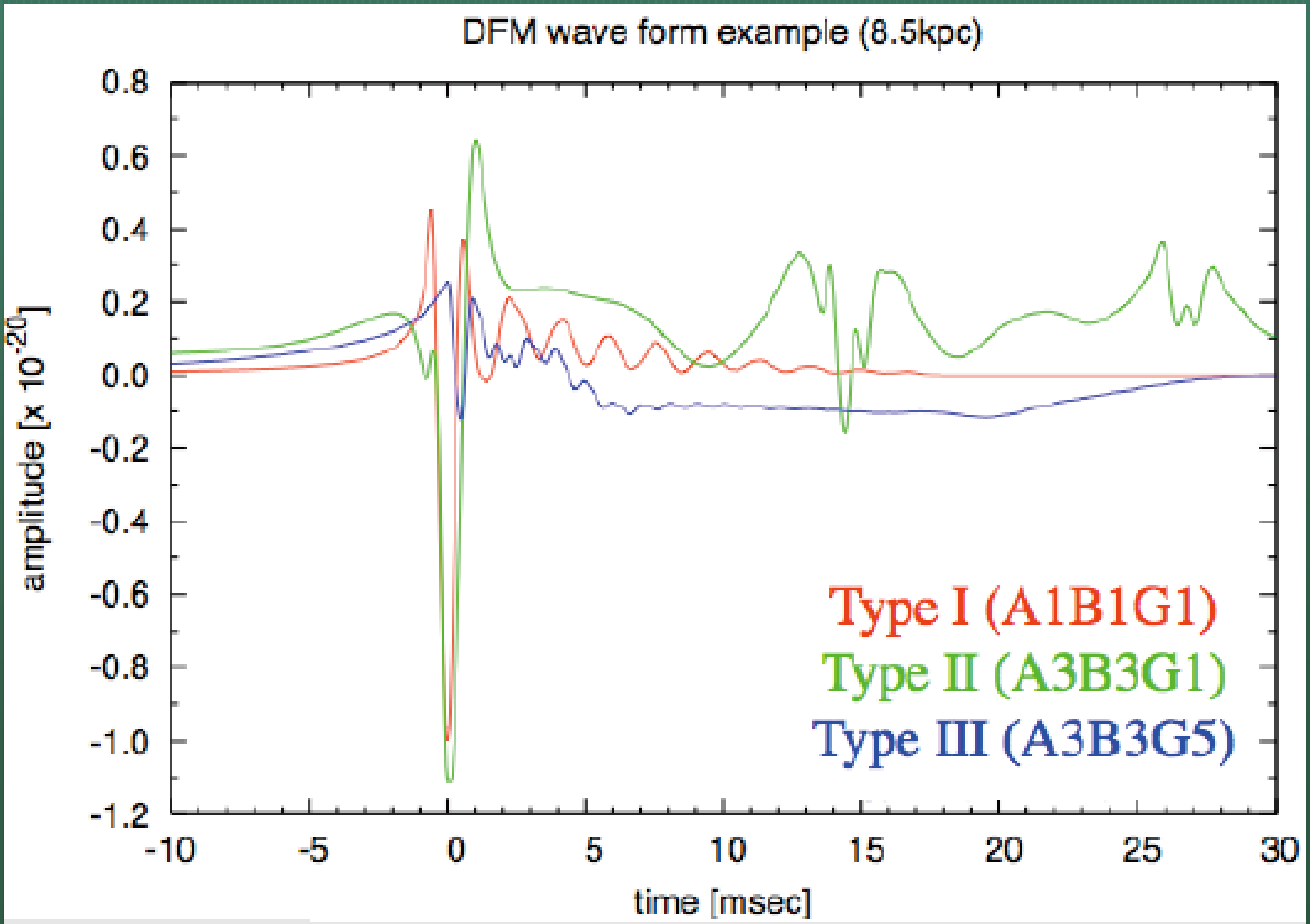
<u>GW generation source in SN</u>

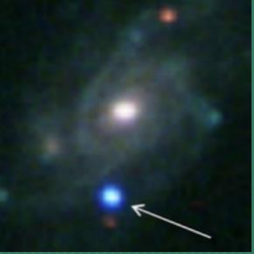
- Core collapse and bounce.
- Convection current around primodial NS
- Unstable rotation of a bar structure in Primodial NS's
- R-mode instability
- Anistropic neutrino radiation.



Calculated GW wave forms from models

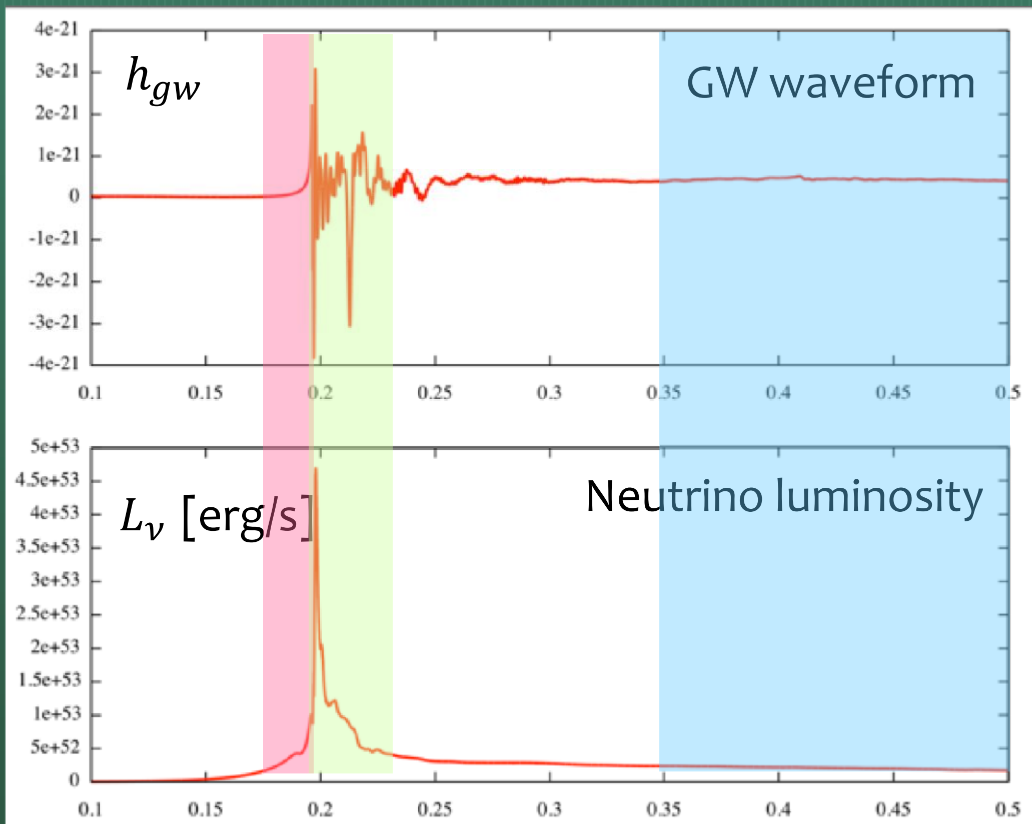
Principally, its difficult to precisely calculate wave forms from SNs
GWs are burst like. Its duration time is only several decades seconds.





GWs and Neutrinos from CoG

Strong GW from core bounce?	Characteristics of prompt convection phase?	Characteristics of SASI phase?



Yokozawa in JPS 2015 Spring

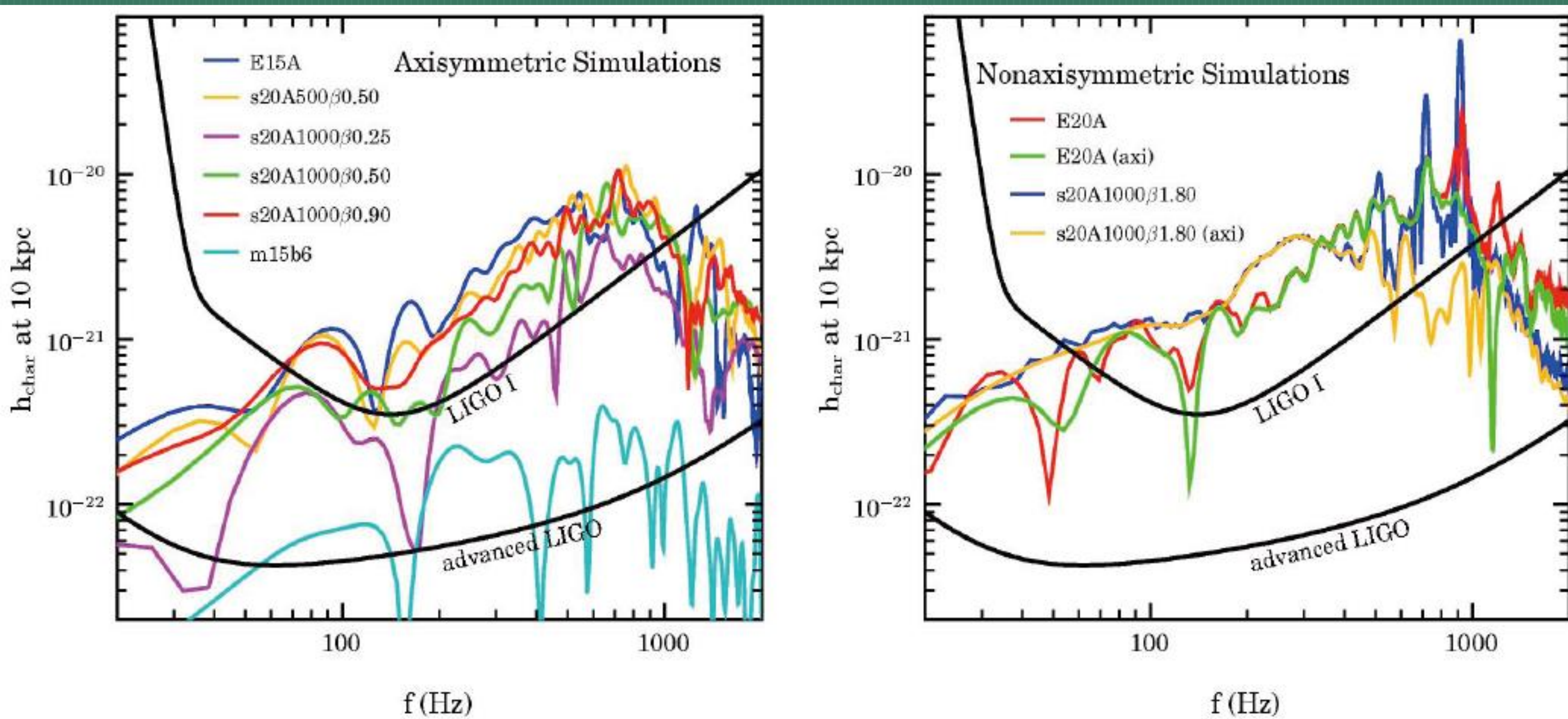
Rapid core rotation model
progenitor mass:11.2M
Suwa et. al. 2013

2D Numerical simulation
Bring us the inner core information.

Identify the characteristics waveform for each phase

- Inner core information by GW and Neutrino
- Understand the mechanism from concurrent analysis
- Time domain astronomy with multi-messanger

Supernovae in our or near galaxy



By C.D. Otto

Expected ? GW Strain Amplitude

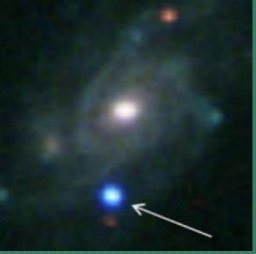
GW Amplitude from SNs st the enter of Galaxy (8.5 kpc)

$$h_{rss} = 10^{-22} \sim 10^{-21}\ [1/\sqrt{\mathrm{Hz}}] \qquad h_{rss} = \sqrt{\int |h(t)|^2 dt}$$

GW Energy among explosion
(is just estimated from a fraction of the total explosion energy)

$$E_{GW} = 10^{45} \sim 10^{49}\ [\mathrm{erg}]$$
$$(10^{-9} \sim 10^{-5} \text{ of total energy } 10^{53} [\mathrm{erg}])$$

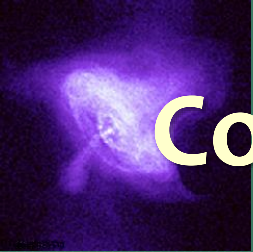
99% energy will be used by neutrino



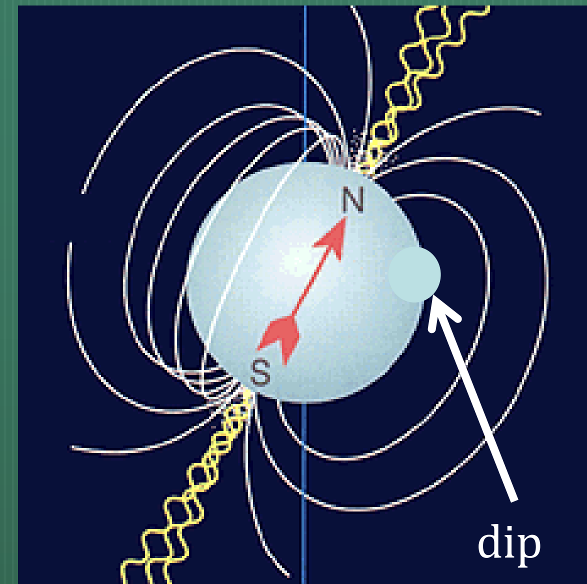
How to Analyze Burst-like GWs

- Excess Power Filtering
- Alternative Linear Fitting Filtering
- Wavelet Filter
- Hilbert-Huang Transformation

- Assuming general burst-like pulse shape (Sine-Gaussian, Pulse - Correlation)



Continuous GWs from Rotating NSs



Properties

Wave form is sinusoidal.
Small amplitude, but continuous.
So the signals can be enlarged by data accumulation.

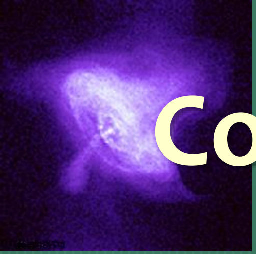
Amplitude

$$h = \frac{16\pi G}{c^4} \frac{\epsilon\, Q_{zz}\, f_{GW}^2}{D}$$

$$= 4 \times 10^{-27} \left(\frac{\epsilon}{10^{-7}}\right) \left(\frac{Q_{zz}}{10^{45}\,[\mathrm{g\ cm^2}]}\right) \left(\frac{f_{GW}}{100\ [\mathrm{Hz}]}\right) \left(\frac{1\ \mathrm{kpc}}{D}\right) [1/\sqrt{\mathrm{Hz}}]$$

ϵ : non axis symmetry (corresponds to 1mm dip on NS (radius ~ 10km))
Q_{zz} : moment of Inertia in Z

$$\epsilon = \frac{Q_{xx} - Q_{yy}}{Q_{zz}} \sim 1 \times \left(\frac{\epsilon}{10^{-7}}\right) \left(\frac{R_{NS}}{10\ [\mathrm{km}]}\right)$$



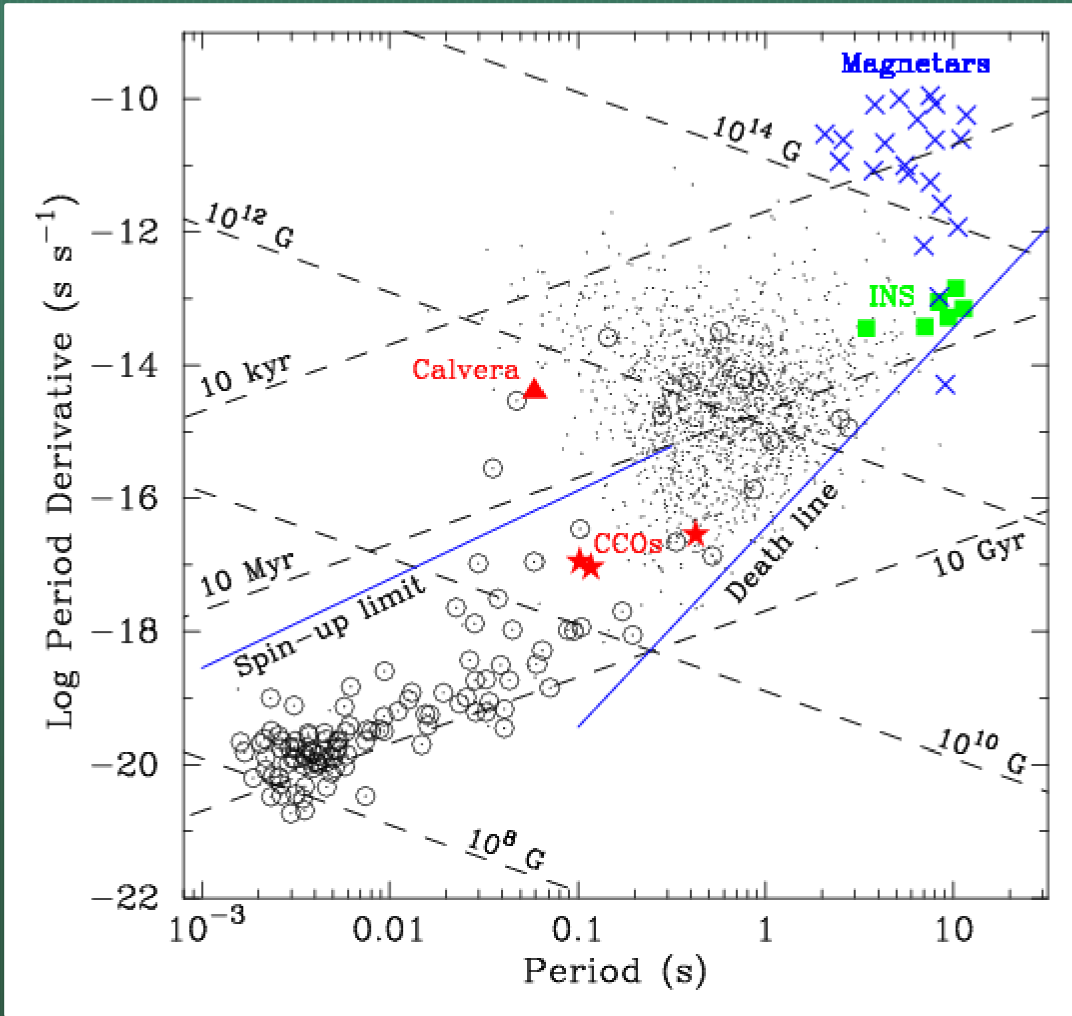
Continuous GWs from Rotating NSs

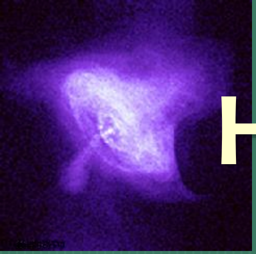
- For crab pulsar assuming all spin-down is caused by GW radiation. (The spin-down can be caused by EMW radiation.)

$$h < 1.4 \times 10^{-24}$$

$$h_{spindown} = \frac{5GI_{zz}}{2c^3 D_L^2}\left(\frac{\dot{P}}{P}\right)$$

- Each cluster might show that there are several types of NS relating with the strength of the magnetic fields ?





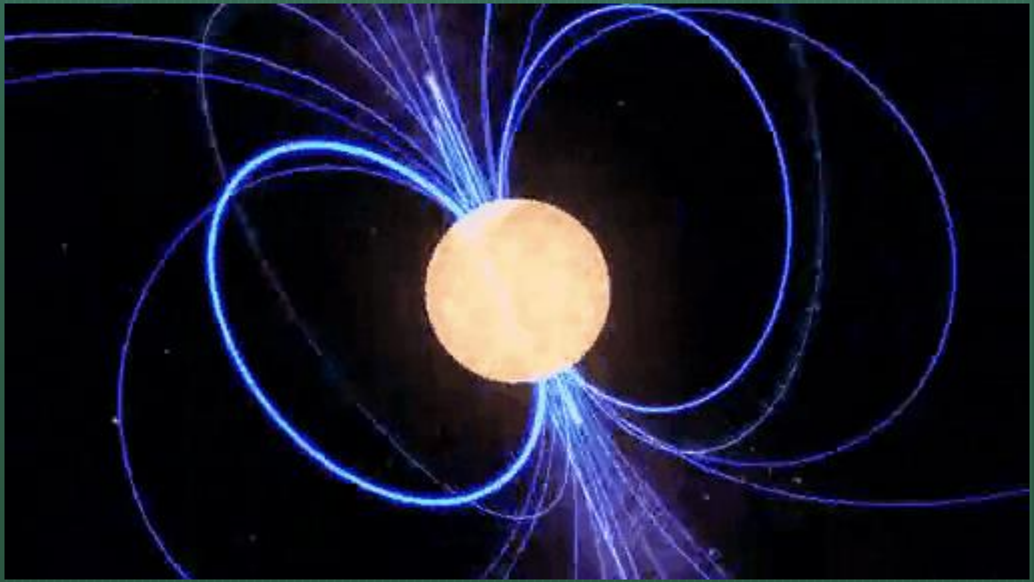
How to Analyze Continuous GWs

Matched Filtering

Wave form is sinusoidal and well known.
However we should note
: Glitches sometimes happen and change frequency,
: Requires a several years accumulation,
: The effect of rotation of the Earth,
: NS spin down and NS motion.

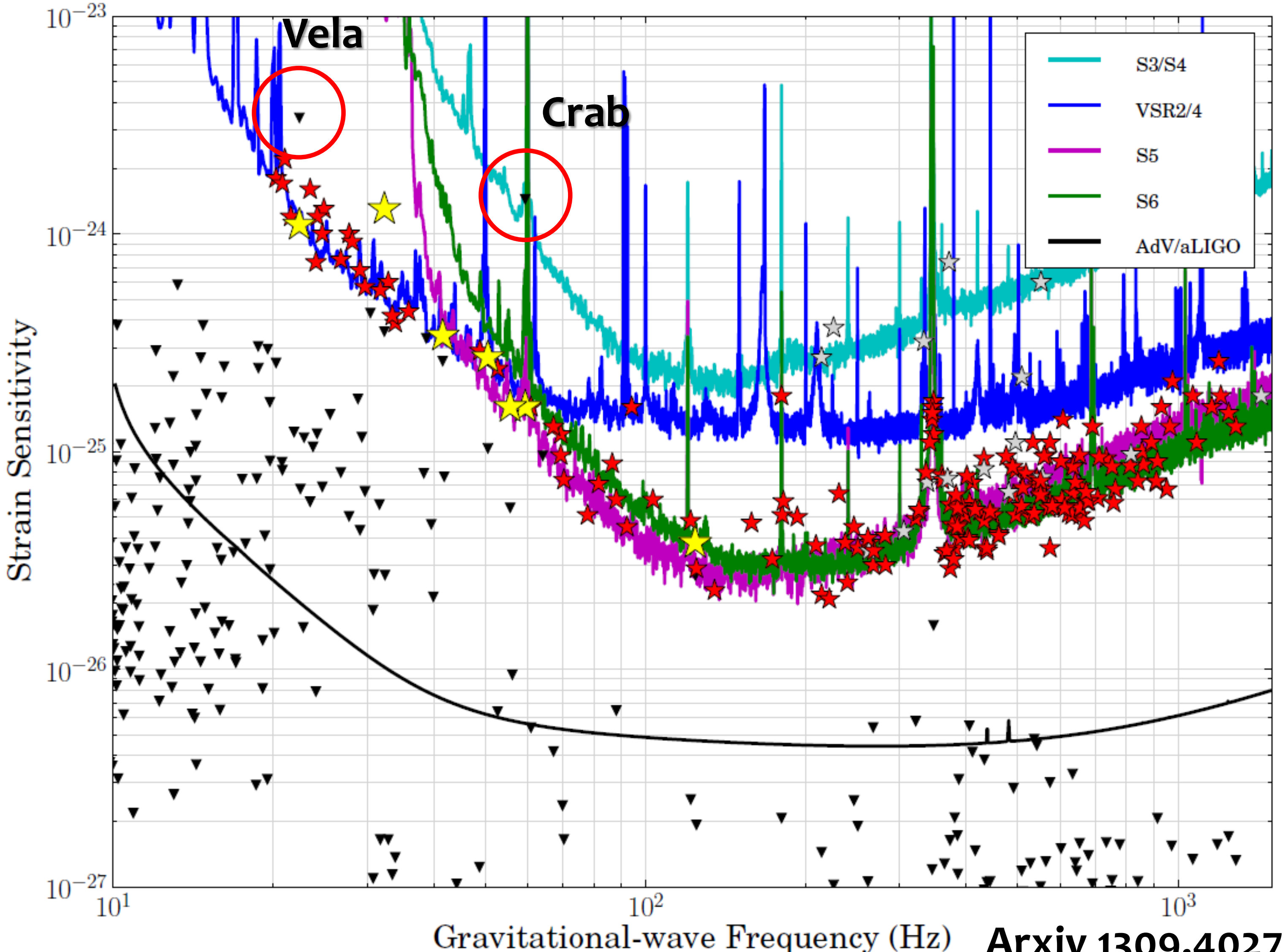
Exciting possibility

Can we find near unknown NSs whose pulse don’t sweep on the Earth ?



http://holographicgalaxy.blogspot.jp/2013/10/pulsars-are-unknown-dark-matter.html

GW upper limits from Pulsars



Arxiv 1309.4027

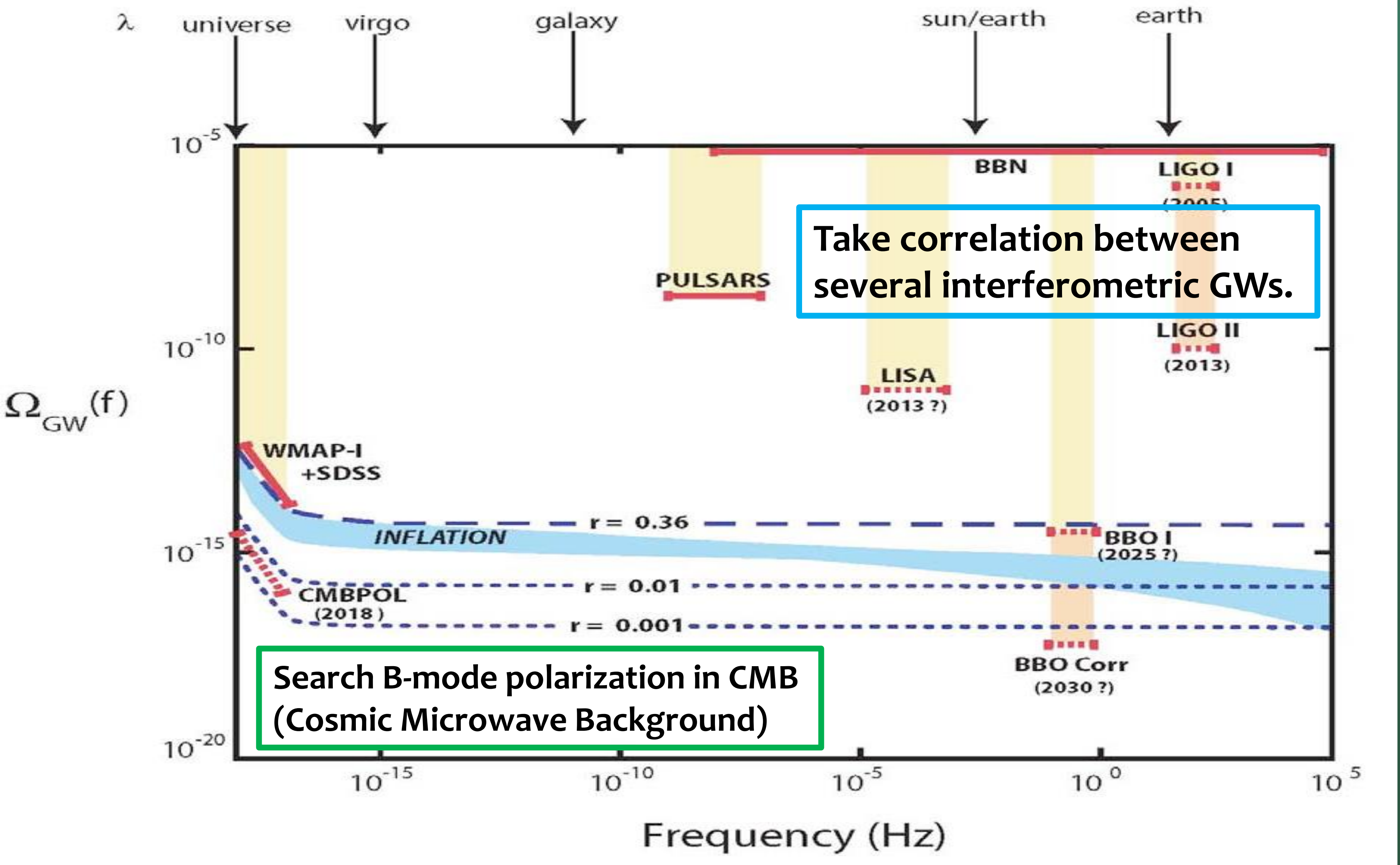
Stochastic GWs

- Properties -

- Stochastic GWs come from all directions.
- One origin is enlarged space-time quantum fluctuation before inflation.
- Other are accumulation of many GWs from known event mechanism (NS-NS, BH-BH, CW, SNe).
- Pop III stars’ binary coalescence might contribute.

Stochastic GWs

- How to detect -

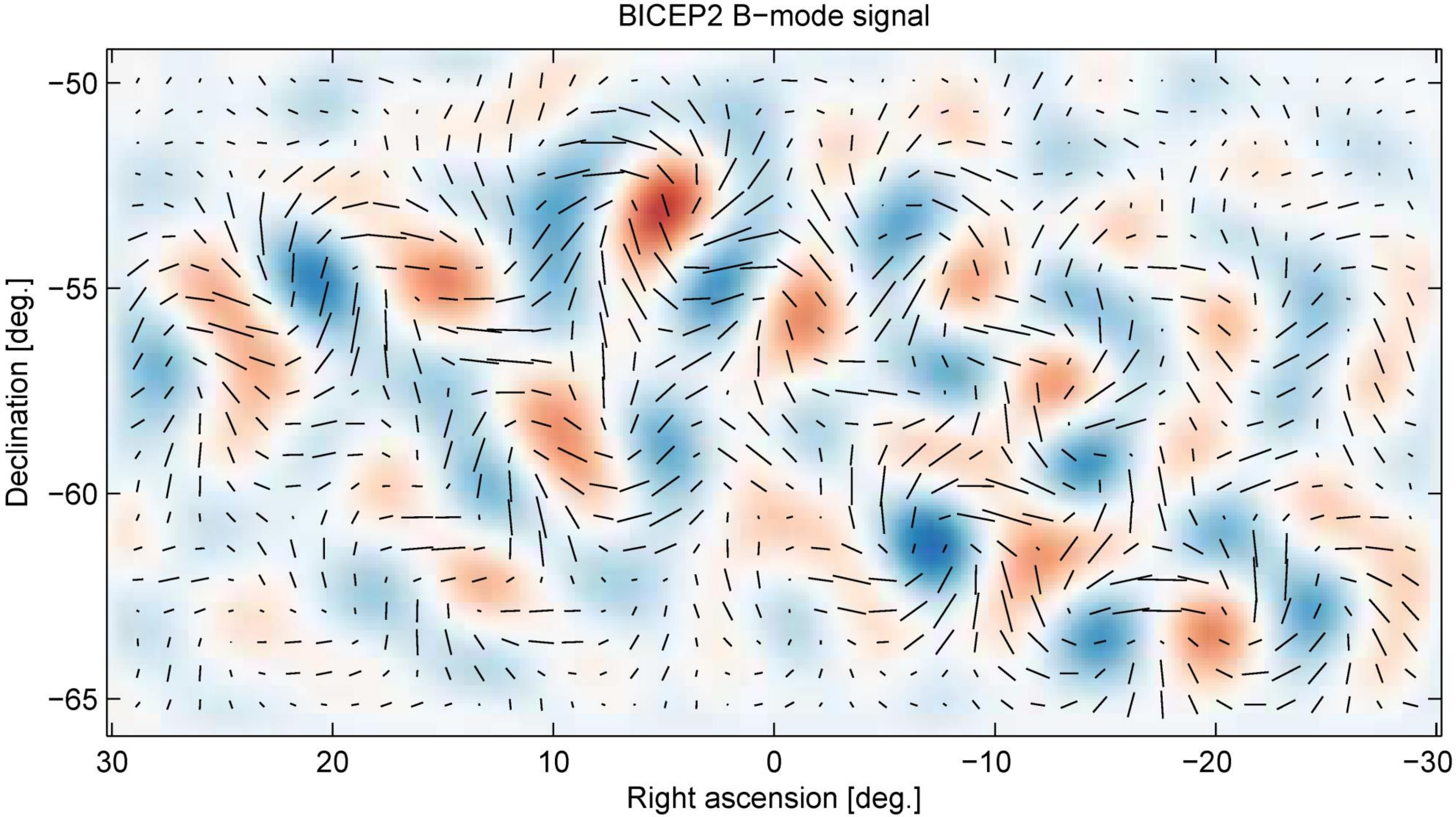


Take correlation between several interferometric GWs.

Search B-mode polarization in CMB (Cosmic Microwave Background)

Primordial GWs detection

- http://bicepkeck.org/visuals.html -



Primordial GWs detection

- http://bicepkeck.org/visuals.html -