略解がある問題については，各自自己採点・修正確認し理解を深めて下さい．

問 1

略解 極座標での微分演算子

極座標系の単位ベクトル

$$\begin{aligned} \vec{e}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ \vec{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \\ \vec{e}_\phi &= (-\sin\phi, \cos\phi, 0), \end{aligned}$$

ここで，

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} \vec{r} = \begin{pmatrix} \vec{e}_r \\ r\vec{e}_\theta \\ r\sin\theta\vec{e}_\phi \end{pmatrix}. \tag{1}$$

を利用すると，

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = \vec{e}_r\cdot\vec{\nabla}_c, \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} = r\vec{e}_\theta\cdot\vec{\nabla}_c, \\ \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} = r\sin\theta\vec{e}_\phi\cdot\vec{\nabla}_c. \end{aligned}$$

ただし，$\vec{\nabla}_c = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ である．したがって，

$$\vec{\nabla}_c = \vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{2}$$

が得られる．単位ベクトルの微分:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} (\vec{e}_r\ \ \vec{e}_\theta\ \ \vec{e}_\phi) = \begin{pmatrix} 0 & 0 & 0 \\ \vec{e}_\theta & -\vec{e}_r & 0 \\ \sin\theta\vec{e}_\phi & \cos\theta\vec{e}_\phi & -\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta \end{pmatrix} \tag{3}$$

に注意すると, 次の計算ができる．

$$\begin{aligned} \Delta &= \vec{\nabla}_c^2 \\ &= \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\ &= \vec{e}_r\frac{\partial}{\partial r}\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta}\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\ &\quad + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta}\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta}\right) + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial \phi^2}\right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \end{aligned}$$

問 2

略解 (1) (a) $\omega = 2\pi\nu$, $\lambda = \frac{v}{\nu}$, $k = \frac{\omega}{v} = 2\pi\frac{\nu}{v}$.

(b) $u = A\cos(kz - \omega t + \phi)$.

(c) $k\lambda - \omega t + \phi = 2\pi - \omega t + \phi$.

(d) $\boldsymbol{k} = k\boldsymbol{n} = k(0, \sin\theta, \cos\theta)$, $u = A\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi) = A\cos(ky\sin\theta + kz\cos\theta - \omega t + \phi)$.

(2) (a) $r = \sqrt{x^2+y^2+z^2}$ として，$u = \dfrac{B}{r}\cos\{2\pi(r - vt)/\lambda + \phi'\}$.

問 **3**

略解 (1) $x = r\cos\theta$, $y = r\sin\theta$.

(2) $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$.

(3) (a) $|z| = \sqrt{x^2 + y^2} = r$.

(b) $\dfrac{z}{|z|} = \dfrac{x + iy}{\sqrt{x^2 + y^2}} = e^{i\theta}$.

(c) $z^* = x - iy = re^{-i\theta}$.

(d) $\dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2} = \dfrac{e^{-i\theta}}{r}$.

(e) $z^2 = x^2 - y^2 + 2ixy = r^2 e^{2i\theta}$.

注意 z を基準として求める点を図中に明確に示すことが大切です．1 と比較して r の大きさを考えることや，θ の 2 倍や $-\theta$ を適切にコンパスと定規を利用して描きましょう．